Friendship Junior High School
Accelerated Math Program
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A.T.I.M.

Advanced Topics In Mathematics

UNIT 1
Polynomials

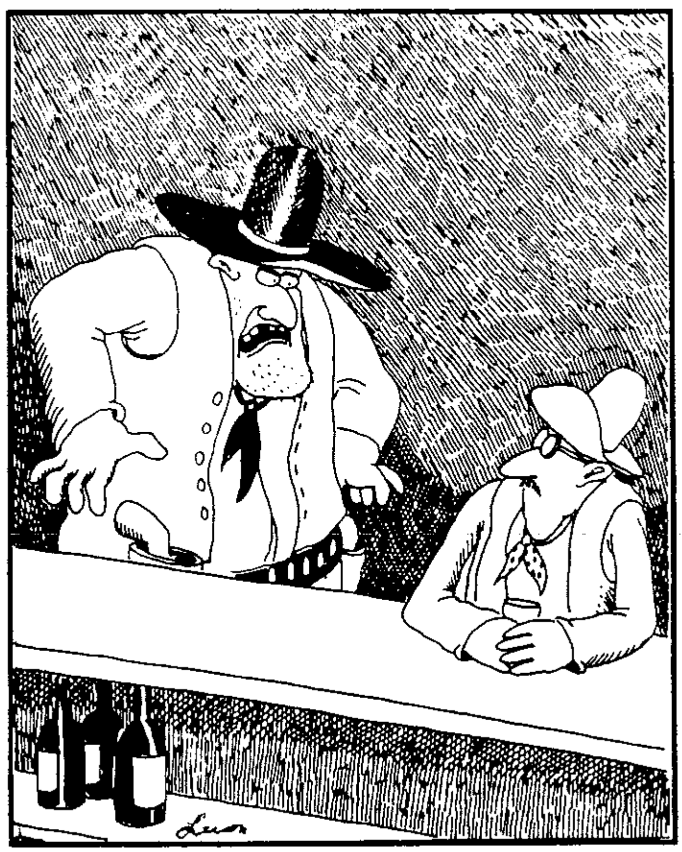
UNIT 2
Systems

UNIT 3
Problem Solving

UNIT 17
Sequence & Series

UNIT 18
Combinations & Permutations

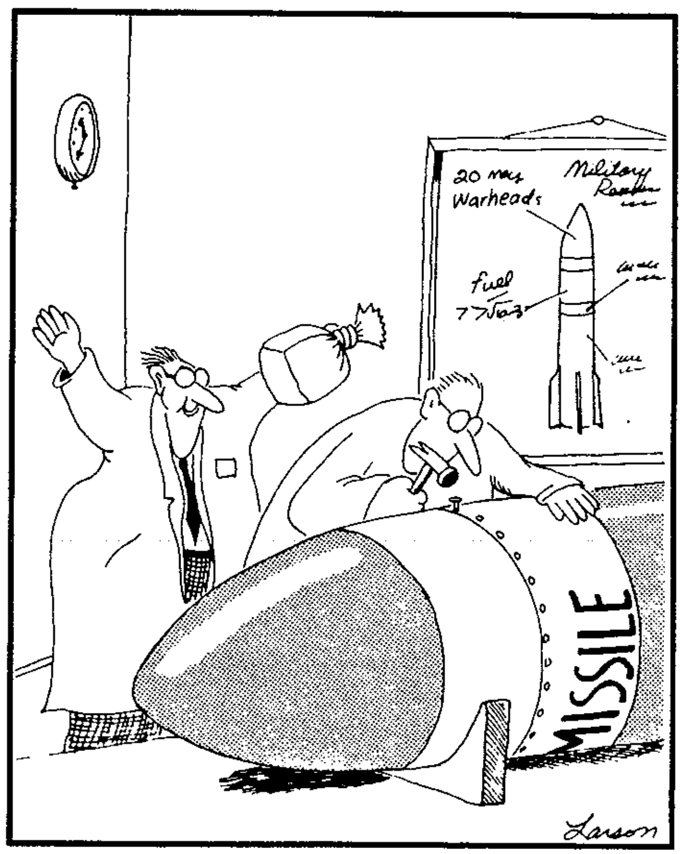
UNIT 19
Mathematics of Chance



"I asked you a question, buddy . . . What's the square root of 5,248?"

UNIT 1

Polynomials



Simplifying Polynomials

DEMONSTRATION 1.1

The following steps for simplifying monomials and polynomials are a review of Algebra I skills.

Combining Like Terms

① $(3a^3)(12ab^2) + 9a^2(10b^2a^2) - (3b)^2(2a)^4$
$(3a^3)(12ab^2) + 9a^2(10b^2a^2) - (9b^2)(16a^4)$
$(36a^4b^2) + (90a^4b^2) - (144a^4b^2) = -18a^4b^2$

② $(3xy^2)^2z^3 - (-4yz)^3x^2y + (-2y)^4(xz)^2z$
$(9x^2y^4)z^3 - (-64y^3z^3)x^2y + (16y^4)(x^2z^2)z$
$(9x^2y^4z^3) - (-64x^2y^4z^3) + (16x^2y^4z^3) = 89x^2y^4z^3$

Dividing Monomials & Negative Exponents

③ $\dfrac{xy^4}{x^6y^{-3}z^{-2}} = \dfrac{y^7z^2}{x^5}$

④ $\dfrac{(2ab)^2(a^3c)^{-2}}{(2b^2c)^{-3}}$

$\dfrac{(4a^2b^2)(a^{-6}c^{-2})}{2^{-3}b^{-6}c^{-3}}$

$\dfrac{4a^{-4}b^2c^{-2}}{2^{-3}b^{-6}c^{-3}} = \dfrac{32b^8c}{a^4}$

Variable Exponents

⑤ $\dfrac{4x^3y^{2n-1}}{-2xy^{n+3}} = -2x^2y^{(2n-1)-(n+3)} = -2x^2y^{n-4}$

Negative Exponents: More Challenging Problems

⑥ $\left(\dfrac{-2x^2y^{-1}}{x^{-3}y^3}\right)^{-2}$

$\left(\dfrac{-2x^5}{y^4}\right)^{-2} = \left(\dfrac{y^4}{-2x^5}\right) = \dfrac{y^8}{4x^{10}}$

⑦ $\left(\dfrac{-3a^2b^{-3}}{a^4b^{-1}}\right)^{-3}$

$\left(\dfrac{-3}{a^2b^2}\right)^{-3} = \left(\dfrac{a^2b^2}{-3}\right)^3$

$\dfrac{-a^6b^6}{27}$

Simplifying Polynomials

DEMONSTRATION 1.1

Distributive Property

⑧ $a^{-2}b\,(a^3b^{-1}+ab^2-2a^{-3}b^{-3})$

$a+a^{-1}b^3-2a^{-5}b^{-2}$

$a+\frac{b^3}{a}-\frac{2}{a^5b^2}$

Squaring A Binomial

⑨ $(3x-5y)^2$

$9x^2-30xy+25y^2$

Product of A Sum & Difference

⑩ $(2n+3m)(2n-3m)$

$4n^2-9m^2$

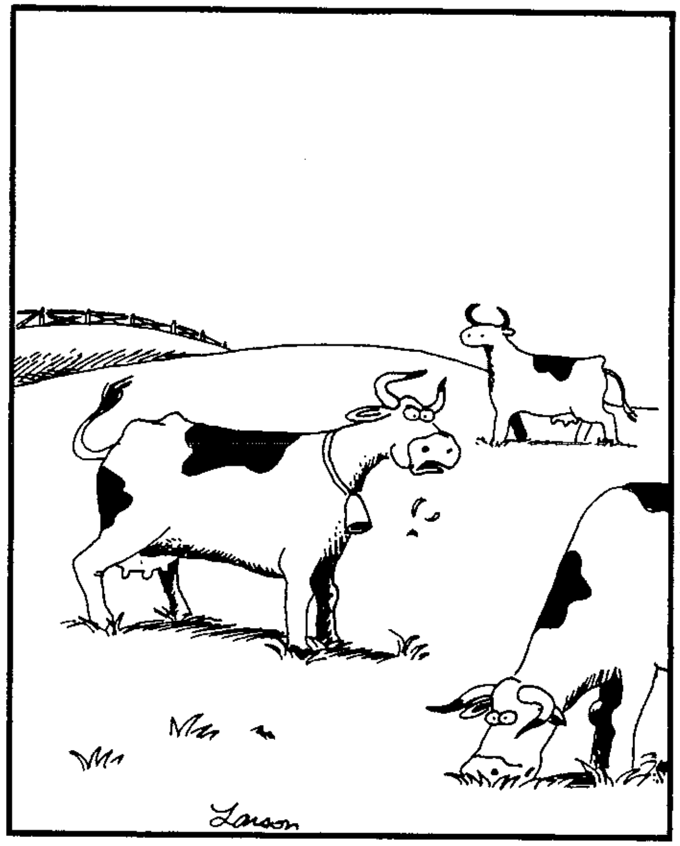
Special Products with Variable Coefficients

⑪ $(2x^{3n}+3y^{n+2})^2$

$4x^{6n}+12x^{3n}y^{n+2}+9y^{2n+4}$

⑫ $(a^{3x+1}+3b^{x-2})(a^{3x+1}-3b^{x-2})$

$a^{6x+2}-9b^{2x-4}$



"Hey, wait a minute! This is grass! We've been eating grass!"

Simplifying Polynomials

PROBLEM SET 1.1

Simplify Each Monomial or Polynomial Expression:

1. $(5a)(6a^2b)(3ab^3) + (4a^2)(3b^3)(2a^2b)$
2. $(5mn^2)(m^3n)(-3p^2) + (8np)(3mp)(m^3n^2)$
3. $(2xy^2)^3 + (2xy^2)^2(6xy^2)$
4. $(3a)(a^2b)^3 + (2a)^2(-a^5b^3)$
5. $\dfrac{-2a^3b^6}{24a^2b^2}$
6. $\dfrac{3m^{-4}}{4^{-1}m^{-2}}$
7. $\dfrac{14a^{-2}bc^{-3}}{6a^3b^4c^{-1}}$
8. $\dfrac{26x^{-3}y^4z^{-5}}{-18x^4y^{-3}z^{-2}}$
9. $\dfrac{(-2r^3)^2(r^{-2})^{-1}}{(r^2)^{-3}}$
10. $\dfrac{(-3x^{-2})^{-1}(x^2)^3}{-(-2x)^2}$
11. $\dfrac{x^{2n}}{x^{2n-3}}$
12. $\dfrac{3n^{2x-4}}{n^{x+3}}$
13. $\left(\dfrac{-3y^4}{2y^2}\right)^{-2}$
14. $\left(\dfrac{3}{2x^{-2}}\right)^{-1}$
15. $\left(\dfrac{x}{y^{-1}z^2}\right)^{-1}$
16. $\left(\dfrac{-2y^3}{x^{-2}y^{-1}}\right)^{-2}$
17. $m^{-3}(m^2+m^4-m^{-1})$
18. $a^3(a^{-2}+a^{-5}+a)$
19. $4a^{-1}b^2(a^2b^{-1}+3a^3b^{-2}+4^{-2}ab^{-1})$
20. $y^2x^{-3}(yx^4+y^{-1}x^3+y^{-2}x^2)$
21. $(2n-3m)(2n+3m)$
22. $(3x-2y)^2$
23. $(2n^{x+1}+3m^{3x})^2$
24. $(a^{n+1}+3b^{4n})(a^{n+1}-3b^{4n})$
25. $(2x^{3a-1}+4y^a)(2x^{3a-1}-4y^a)$
26. $(5x^{3n+1}-2y^{n-2})^2$

Factoring Perfect Cubes

DEMONSTRATION 1.2

The following is a review and extension of Algebra I factoring methods. New methods are included for factoring the sum and difference of perfect cubes, the factor theorem, and factoring the sum and difference of like odd powers.

Greatest Common Factor

① $6a^2b^3-18a^3b^2 = 6a^2b^2(b-3a)$

② $14n^3m+8nm^2 = 2nm(7n^2+4m)$

Difference of Perfect Squares

③ $a^2-4b^2 = (a+2b)(a-2b)$

④ $3x^2-27y^4$

$3(x^2-9y^4) = 3(x+3y^2)(x-3y^2)$

Perfect Square Trinomial

⑤ $n^2+8n+16 = (n+4)^2$

⑥ $2x^2-12xy+18y^2$

$2(x^2-6xy+9y^2)=2(x-3y)^2$

Trinomial: ax^2+bx+c

⑦ $3x^2+10xy+8y^2$

$3x^2+6xy+4xy+8y^2$

$3x(x+2y)+4y(x+2y)=(x+2y)(3x+4y)$

Grouping

⑧ $x^6-x^4y^2+y^6-x^2y^4$

$x^4(x^2-y^2)+y^4(y^2-x^2)$

$x^4(x^2-y^2)-y^4(x^2-y^2)$

$(x^2-y^2)(x^4-y^4)$

$(x+y)(x-y)(x^2+y^2)(x+y)(x-y)$

Quantities

⑨ $(4a+3b)^2-(a-b)^2$

$[(4a+3b)+(a-b)][(4a+3b)-(a-b)]$

$(5a+2b)(3a+4b)$

Sum or Difference of Perfect Cubes

⑩ x^3-y^3

$(x-y)(x^2+xy+y^2)$

⑪ x^3+y^3

$(x+y)(x^2-xy+y^2)$

⑫ $27a^3+8b^6$

$(3a+2b^2)(9a^2-6ab^2+4b^4)$

continued

Factoring Perfect Cubes

DEMONSTRATION 1.2

(13) $2a^6 - 128b^9$

$2(a^6 - 64b^9)$

$2(a^2 - 4b^3)(a^4 + 4a^2b^3 + 16b^6)$

Factor Theorem

(14) $x^3 + x^2 - 8x - 12$

Factors of -12
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

Try factors until the expression equals 0

$(1)^3 + (1)^2 - 8(1) - 12 = -18$
$(-1)^3 + (-1)^2 - 8(-1) - 12 = -4$
$(2)^3 + (2)^2 - 8(2) - 12 = -16$
$(-2)^3 + (-2)^2 - 8(-2) - 12 = 0$

Since (-2) works, $(x+2)$ is a factor. To find the other factor, use polynomial division:

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x+2 & x^3 + x^2 - 8x - 12 \\ & \underline{x^3 + 2x^2} \\ & \quad -x^2 - 8x \\ & \quad \underline{-x^2 - 2x} \\ & \qquad -6x - 12 \\ & \qquad \underline{-6x - 12} \end{array}$$

$(x+2)(x^2 - x - 6)$
$(x+2)(x+2)(x-3)$

(15) For extra practice using the factor theorem, try:

$x^3 + 4x^2 + x - 6$

Sum or Difference of Like Odd Powers

(16) $x^5 + y^5$
$(x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

(17) $x^5 - y^5$
$(x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

(18) $n^7 - 128$

$n^7 - 2^7$

$(n-2)(n^6 + 2n^5 + 4n^4 + 8n^3 + 16n^2 + 32n + 64)$

(19) $32x^5 - b^{10}$

$(2x)^5 - (b^2)^5$

$(2x - b^2)[(2x)^4 + (2x)^3(b^2) + (2x)^2(b^2)^2 + (2x)(b^2)^3 + (b^2)^4]$

$(2x - b^2)(16x^4 + 8x^3b^2 + 4x^2b^4 + 2xb^6 + b^8)$

Factoring Perfect Cubes

PROBLEM SET 1.2

Factoring review:

1. $5x^2y - 10xy^2$
2. $-15x^2 - 5x$
3. $r^2 - 9$
4. $x^2 - 49$
5. $k^2 + 12k + 36$
6. $4n^2 - 20np + 25p^2$
7. $3y^2 + 5y + 2$
8. $4x^2 + 11x + 6$
9. $(2a+b)^2 - (a-3b)^2$
10. $(3x-2y)^2 - (2x+y)^2$
11. $a^2b^2 - a^4 + a^2b^2 - b^4$
12. $x^8n^4 - 16x^8 - n^4 + 16$

Factor the sum or difference of cubes:

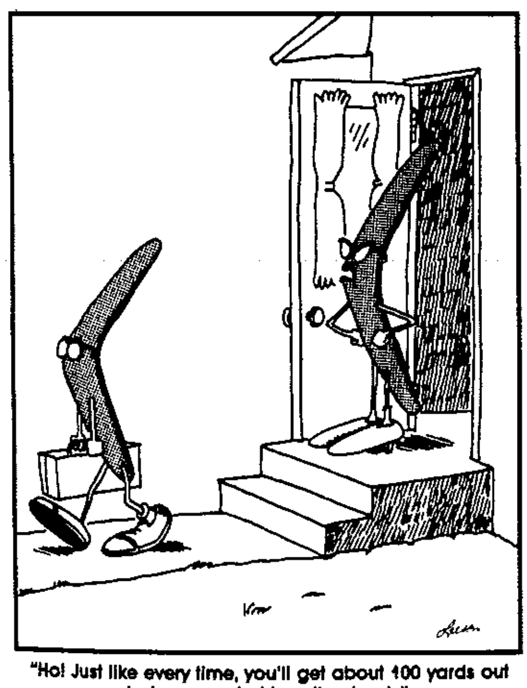
13. $x^3 - y^3$
14. $n^3 + 27$
15. $8b^3 + 27c^3$
16. $2y^3 - 16x^3$
17. $y^6 - n^3m^9$
18. $64x^3 + 27y^6$

Factor completely:

19. $16x^4 - 196y^4$
20. $(a+b)^2 - b^2$
21. $a^5 - a^3b^2 - a^2b^3 + b^5$
22. $a^6 - a^3b^3 - b^6 + a^3b^3$

Factor theorem and like odd powers:

23. $x^3 + 9x^2 + 23x + 15$
24. $x^3 - 2x^2 - 5x + 6$
25. $n^3 - 2n^2 - 9n + 18$
26. $x^7 + y^7$
27. $a^5 - 243b^5$
28. $x^{15} + 32y^{10}$



"Ho! Just like every time, you'll get about 100 yards out before you start heading back."

Grouping Polynomials

DEMONSTRATION 1.3

Sometimes a four term polynomial cannot be grouped 2-and-2.
Many of these polynomials require "creative" grouping.

Factor Completely

① x^3+2x^2-x-2
$x^2(x+2)-1(x+2)$
$(x+2)(x^2-1)$
$(x+2)(x+1)(x-1)$

Order Change / 3-1

② $a^2+4ab-9x^2+4b^2$
$(a^2+4ab+4b^2)-9x^2$
$(a+2b)^2-9x^2$
$(a+2b+3x)(a+2b-3x)$

Special Quantities

③ $14aby+14amy+7b^2y-7m^2y$
$7y(2ab+2am+b^2-m^2)$
$7y\left[2a(b+m)+(b-m)(b+m)\right]$
$7y(b+m)(2a+b-m)$

Order Change / Factor Completely

④ $m^3-3m^2a+3ma^2-a^3$
$m^3-a^3-3m^2a+3ma^2$
$(m-a)(m^2+ma+a^2)-3ma(m-a)$
$(m-a)(m^2+ma+a^2-3ma)$
$(m-a)(m^2-2ma+a^2)$
$(m-a)(m-a)^2 = (m-a)^3$

Quadratic Form

⑤ n^4-10n^2+9
$(n^2)^2-10n^2+9$
$(n^2-9)(n^2-1)$
$(n+3)(n-3)(n+1)(n-1)$

Additional Practice

2a $4xy-n^2+4x^2+y^2$
$(4x^2+4xy+y^2)-n^2$
$(2x+y)^2-n^2$
$(2x+y+n)(2x+y-n)$

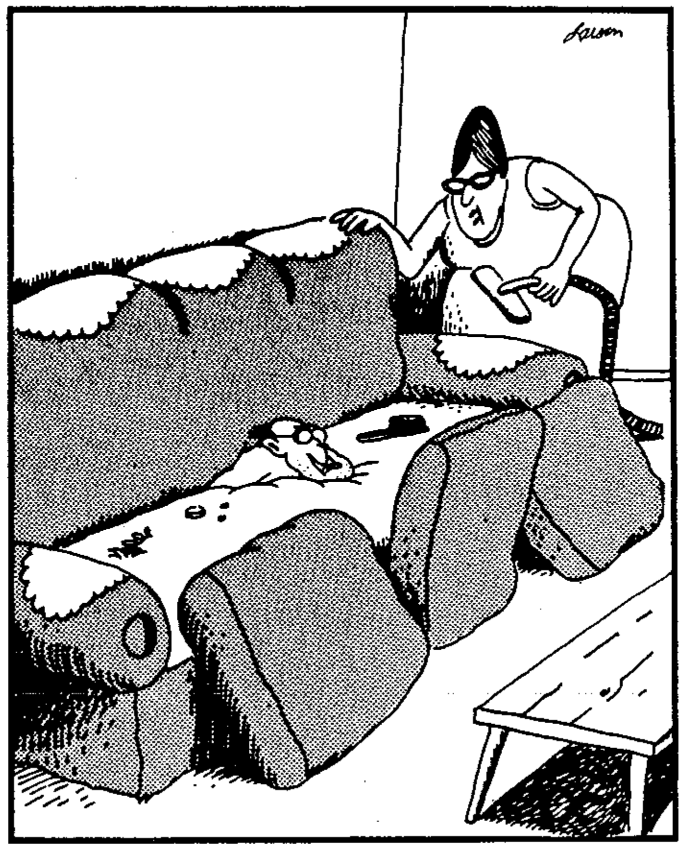
4a $8x^3+14x^2+1+7x$
$8x^3+1+14x^2+7x$
$(2x+1)(4x^2-2x+1)+7x(2x+1)$
$(2x+1)(4x^2-2x+1+7x)$
$(2x+1)(4x^2+5x+1)$
$(2x+1)(4x^2+4x+x+1)$
$(2x+1)\left[4x(x+1)+1(x+1)\right]$
$(2x+1)(x+1)(4x+1)$

Grouping Polynomials

PROBLEM SET 1.3

Factor completely:

1. $k^3+4k^2-9k-36$
2. $a^2x-b^2x+a^2y-b^2y$
3. $x^2-y^2+4y-4x$
4. $a+b+3a^2-3b^2$
5. $b^2-y^2-p^2-2yp$
6. $a^2-9+2ab+b^2$
7. $2ab+2am+b^2-m^2$
8. $3pq+3ps+q^2-s^2$
9. $16m^3-2$
10. $3r+81r^4$
11. $x^3+y^3-x^2y-xy^2$
12. $t^3+125+5t^2+25t$
13. x^4-13x^2+36
14. y^4-14y^2+45
15. $r^2-rt-rt^2+t^3$
16. $8ax-6x-12a+9$
17. $10x^2-14xy-15x+21y$
18. $a^4-12a^3b+24a^2b^2-8ab^3$
19. $a^4-16a^2+3a^3-48a$
20. $x^3y-3x^2y-6xy+8y$



"Andrew! So that's where you've been! And good heavens! . . . There's my old hairbrush, too!"

Polynomial Division & Synthetic Division

DEMONSTRATION 1.4

Polynomial division and synthetic division are alternate methods for dividing polynomials.

Polynomial Division:

① $(n^3-3) \div (n+2)$

$$\begin{array}{r|l}
 & n^2-2n+4-\frac{11}{n+2} \\
\hline
n+2 & n^3 \qquad\qquad -3 \\
 & \underline{n^3+2n^2} \\
 & \quad -2n^2 \\
 & \quad \underline{-2n^2-4n} \\
 & \qquad\quad 4n-3 \\
 & \qquad\quad \underline{4n+8} \\
 & \qquad\qquad -11
\end{array}$$

② $(3a^4-2a^2+a-4)(a^2+2a-1)^{-1}$

$$\begin{array}{r|l}
 & 3a^2-6a+13+\frac{-31a+9}{a^2+2a-1} \\
\hline
a^2+2a-1 & 3a^4 \qquad -2a^2+a-4 \\
 & \underline{3a^4+6a^3-3a^2} \\
 & \quad -6a^3+a^2+a \\
 & \quad \underline{-6a^3-12a^2+6a} \\
 & \qquad\quad 13a^2-5a-4 \\
 & \qquad\quad \underline{13a^2+26a-13} \\
 & \qquad\qquad -31a+9
\end{array}$$

Synthetic Division:

③ $(2x^4-5x^3-10x+8)(x-3)^{-1}$

$$\begin{array}{r|rrrr|r}
3 & 2 & -5 & 0 & -10 & 8 \\
 & & 6 & 3 & 9 & -3 \\
\hline
 & 2 & 1 & 3 & -1 & 5
\end{array}$$

$2x^3+x^2+3x$
$-1+5/x-3$

④ $(12n^4-2n^3-3) \div (2n+1)$
$(6n^4-n^3-3/2) \div (n+1/2)$

$$\begin{array}{r|rrrr|r}
-\frac{1}{2} & 6 & -1 & 0 & 0 & -3/2 \\
 & & -3 & 2 & -1 & 1/2 \\
\hline
 & 6 & -4 & 2 & -1 & -1
\end{array}$$

$6n^3-4n^2+2n-1-2/2n+1$

a) Lead coefficient (2) is not "1"... divide by 2

b) Take the opposite of the constant in divisor $-\frac{1}{2}|$

c) Write coefficients in descending order of n: $6 \; -1 \; 0 \; 0 \; -3/2$

d) Bring down first number (6)

e) Multiply $-\frac{1}{2}| \times (6) = -3$ and put in next column... Add $= -4$

f) Multiply $-\frac{1}{2}| \times (-4) = 2$ and put in next column... Add $= 2$

g) Continue process

h) Multiply remainder by 2 due to division in step a)

i) Use result to frame quotient

Polynomial Division & Synthetic Division

PROBLEM SET 1.4

Polynomial division:

① $(80x^2+6x-4) \div (10x-3)$

② $(x^3+4x-4) \div (x+2)$

③ $(x^4+4)(x^2-2x+2)^{-1}$

④ $(a^4-3a^2+2)(a^2+a-1)^{-1}$

Synthetic division:

⑤ $(2x^3-3x^2+3x-4) \div (x-2)$

⑥ $(2a^3+a^2-2a+3) \div (a+1)$

⑦ $(4x^4-5x^2+2x+3)(2x-1)^{-1}$

⑧ $(4y^4-5y^2-8y-10)(2y-3)^{-1}$

⑨ $(x^5+32) \div (x+2)$

⑩ $(6x^3-28x^2+19x+3) \div (3x-2)$

Review Problems

Simplify:

⑪ $\left(\dfrac{-x^2y^{-3}z}{2x^{-1}y^2z^3}\right)^{-3}$

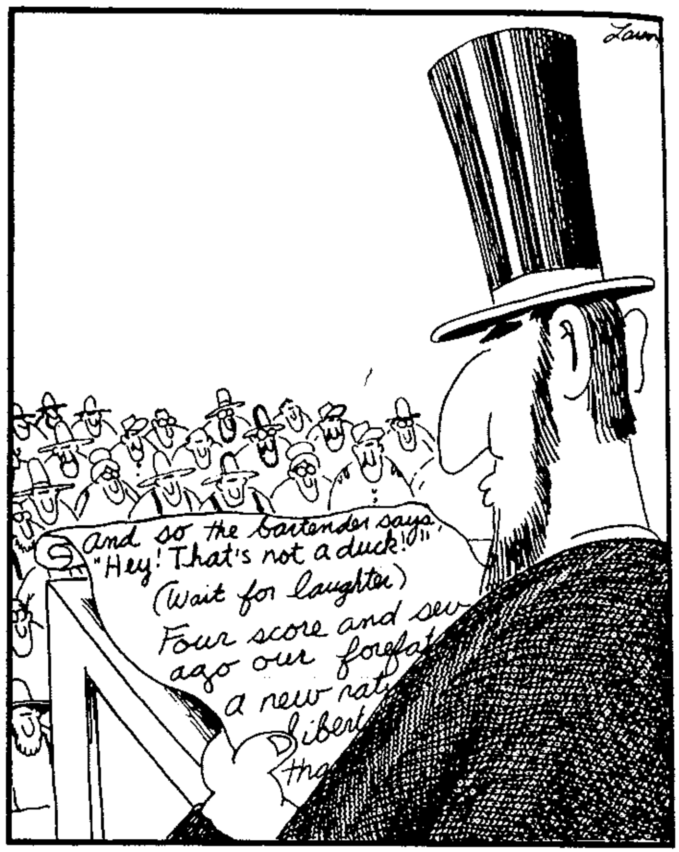
⑫ $(4n^{4x}-3m^{x-5})^2$

Factor completely:

⑬ $a^7-a^3b^4+8b^7-8a^4b^3$

⑭ $2x^3+6xy^2-2y^3-6x^2y$

⑮ $x^2-1-8xy+16y^2$



Polynomials

UNIT 1 REVIEW & PRACTICE

Simplify:

1. $(5a)(6a^2b)(3ab^3)+(2a)^2(3b^3)(2a^2b)$
2. $\dfrac{-9x^{-2}y^3z^{-1}}{3x^{-4}yz^2}$
3. $\left(\dfrac{2y^{-2}}{-2}\right)^{-1}\left(\dfrac{m^2n}{y}\right)^{-2}$
4. $\dfrac{6n^{2x+3}}{3n^{x-4}}$
5. $x^{-1}y^2(x^{-3}y-xy^{-3}+x^2y^{-4})$
6. $(4x^{a+2}-3y^{3a})^2$
7. $(x^{n+1}+y^{2n-1})(x^{n+1}-y^{2n-1})$

Factor completely:

8. $12n^2-7n-10$
9. $(3a-4b)^2-(a+3b)^2$
10. $5x^3y+40y^4$
11. $x^5-x^3y^2+27y^5-27x^2y^3$
12. $x^2-9-6xy+9y^2$
13. $m^4n+m^2n-mn^4-mn^2$
14. $3x^3+9x^2+9x+3$

Polynomial division:

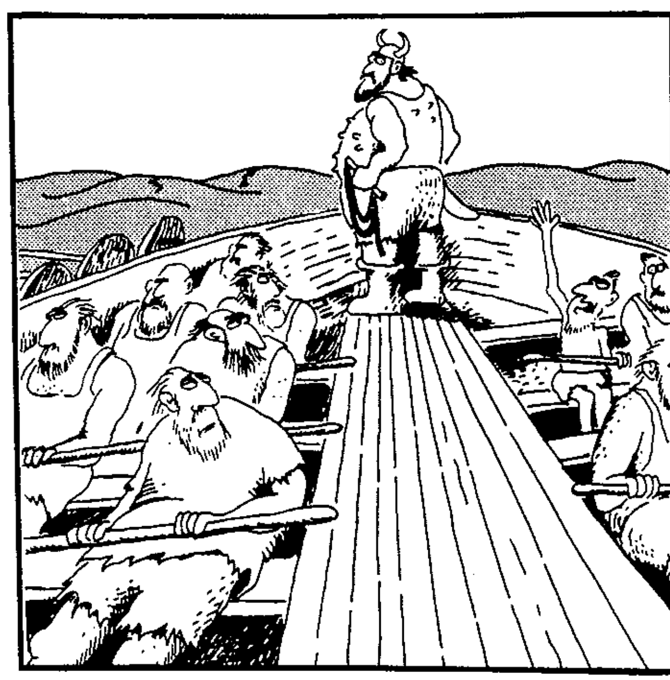
15. $(4n^4-3)(n+2)^{-1}$

Synthetic division:

16. $(x^6-9x^4+2x^3-18x+12)(x+3)^{-1}$
17. $(4x^3+x-3)(2x-1)^{-1}$

Factor theorem and like odd powers:

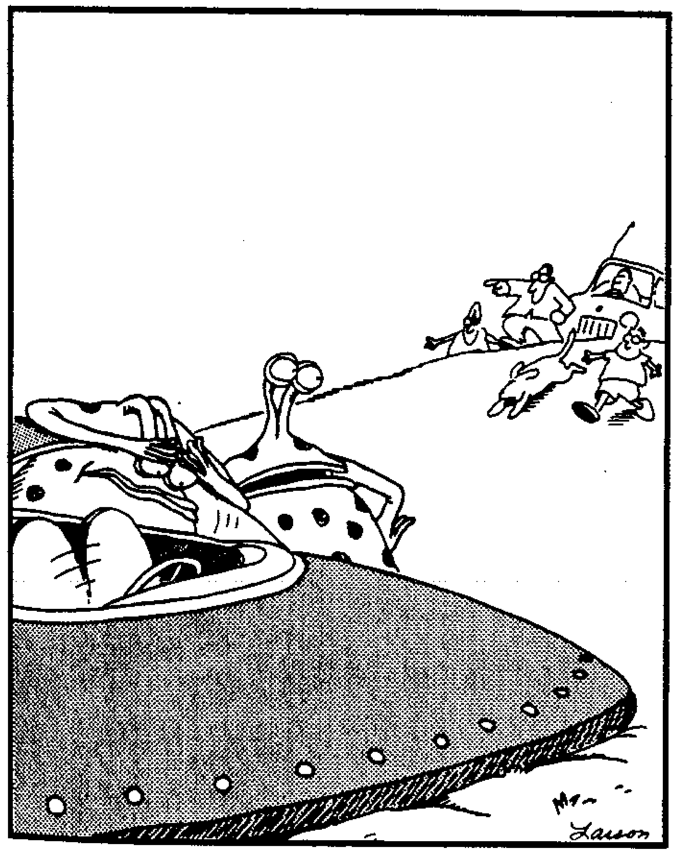
18. $x^3+2x^2-13x+10$
19. $n^3-9n^2+26n-24$
20. x^9+y^9
21. $32x^{10}-y^{20}$



"Yoo-hoo! Oh, yoo-hoo! . . . I think I'm getting a blister."

UNIT 2

Systems



"Well, here they come . . . You locked the keys inside, you do the talkin'."

Second & Third Order Systems

DEMONSTRATION 2.1

Substitution:

① $3x - y = 1 \longrightarrow y = 3x - 1$
$3x + 2y = 16$

$3x + 2(3x-1) = 16$
$3x + 6x - 2 = 16$
$9x = 18$
$x = 2$

$y = 3(2) - 1$
$y = 5$

$(2, 5)$

Elimination:

② $2x + 3y = 2$ mult. by 4
$3x - 4y = -14$ mult. by 3

$8x + 12y = 8$
$\underline{9x - 12y = -42}$
$17x = -34$
$x = -2$

$2(-2) + 3y = 2$
$3y = 6$
$y = 2$

$(-2, 2)$

Systems with 3 Variables:

③ a) $x + 4y + 8z = 7$
b) $6y - 4z = 2$
c) $12z = 3 \rightarrow z = 1/4$

b) $6y - 4(1/4) = 2$
$6y - 1 = 2$
$6y = 3$
$y = 1/2$

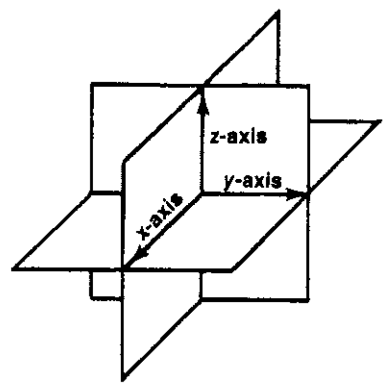
continued

a) $x + 4(1/2) + 8(1/4) = 7$
$x + 2 + 2 = 7$
$x = 3$

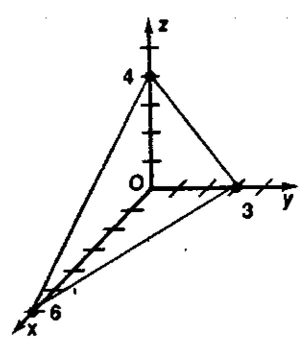
$(3, 1/2, 1/4)$

In 3 dimensions, space is divided into 8 octants.

An equation in 3 variables is shown by a plane with an intercept on each axis.



Example: $2x + 4y + 3z = 12$



View from the first octant.

x-intercept $(6, 0, 0)$

z-intercept $(0, 0, 4)$

y-intercept $(0, 3, 0)$

Second & Third Order Systems

DEMONSTRATION 2.1

4)

a) $2x+3y+2z=14$
b) $4x+2y-z=15$
c) $x+y+3z=8$

$$\begin{array}{rr} a) & 2x+3y+2z=14 \\ c) & \underline{-2x-2y-6z=-16} \\ & y-4z=-2 \end{array}$$

$$\begin{array}{rr} a) & -4x-6y-4z=-28 \\ b) & \underline{4x+2y-z=15} \\ & -4y-5z=-13 \end{array}$$

$y-4z=-2$ mult. by 4
$-4y-5z=-13$

$$\begin{array}{r} 4y-16z=-8 \\ \underline{-4y-5z=-13} \\ -21z=-21 \\ z=1 \end{array}$$

$y-4(1)=-2$
$y-4=-2$
$y=2$

c) $x+(2)+3(1)=8$
$x+5=8$
$x=3$

$(3,2,1)$

5)

a) $x+y+z=0$
b) $2x+3y-3z=8$
c) $x-4y-3z=-11$

$$\begin{array}{rr} a) & -x-y-z=0 \\ c) & \underline{x-4y-3z=-11} \\ & -5y-4z=-11 \end{array}$$

$$\begin{array}{rr} a) & -2x-2y-2z=0 \\ b) & \underline{2x+3y-3z=8} \\ & y-5z=8 \end{array}$$

$-5y-4z=-11$
$y-5z=8$ mult. by 5

$$\begin{array}{r} -5y-4z=-11 \\ \underline{5y-25z=40} \\ -29z=29 \\ z=-1 \end{array}$$

$y-5(-1)=8$
$y+5=8$
$y=3$

a) $x+(3)+(-1)=0$
$x+2=0$
$x=-2$

$(-2,3,-1)$

Second & Third Order Systems

PROBLEM SET 2.1

Solve using substitution:

① $6x - 4y = -6$
$3x + y = 3$

Solve using elimination:

② $8x + 3y = 4$
$4x - 9y = -5$

Solve each system:

③ $x + y + z = -1$
$2x + y = 2$
$-3x = -9$

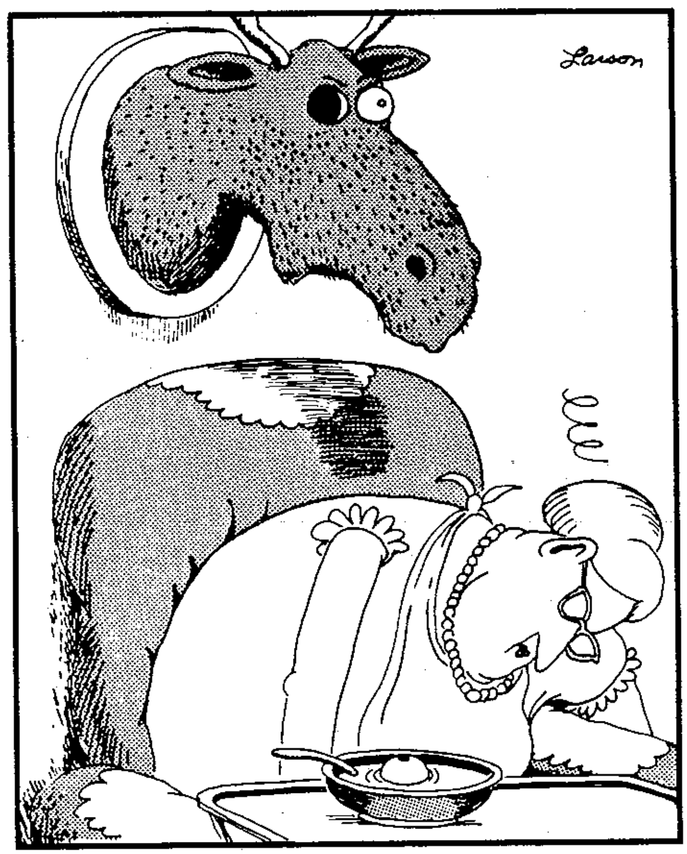
④ $2x + 4y - z = -3$
$y + z = 4$
$2y = -2$

⑤ $x + y + z = -1$
$2x - y + z = 19$
$3x - 2y - 4z = 16$

⑥ $x - 2y + z = 3$
$2x + y - 2z = 31$
$-x + 2y + 3z = -23$

⑦ $2x + 3y + z = 7$
$x + y + z = 4$
$3x + 4y - 2z = 6$

⑧ $x + 8y + 2z = -24$
$3x + y + 7z = -3$
$4x - 3y + 6z = 9$



Determinants & Cramer's Rule

DEMONSTRATION 2.2

Find The Value Of Each Determinant:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

① $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = \begin{matrix} (2)(4)-(3)(1) \\ 8 - 3 = 5 \end{matrix}$

② $\begin{vmatrix} -6 & 7 \\ 0 & -2 \end{vmatrix} = \begin{matrix} (-6)(-2)-(0)(7) \\ 12 - 0 = 12 \end{matrix}$

Use Cramer's Rule To Solve A System:

$$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

$$x = \frac{\begin{vmatrix} C & B \\ F & E \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} A & C \\ D & F \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}}$$

③ $\begin{aligned} 3x - 5y &= -7 \\ x + 2y &= 16 \end{aligned}$

$$x = \frac{\overset{(-14)-(-80)}{\begin{vmatrix} -7 & -5 \\ 16 & 2 \end{vmatrix}}}{\underset{(6)-(-5)}{\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix}}} = \frac{66}{11} \qquad y = \frac{\overset{(48)-(-7)}{\begin{vmatrix} 3 & -7 \\ 1 & 16 \end{vmatrix}}}{11} = \frac{55}{11}$$

$$\boxed{(6, 5)}$$

④ $\begin{aligned} 3x - 5 &= -2y \longrightarrow 3x + 2y = 5 \\ 5x - 6y &= 11 \longrightarrow 5x - 6y = 11 \end{aligned}$

$$x = \frac{\overset{(-30)-(22)}{\begin{vmatrix} 5 & 2 \\ 11 & -6 \end{vmatrix}}}{\underset{(-18)-(10)}{\begin{vmatrix} 3 & 2 \\ 5 & -6 \end{vmatrix}}} = \frac{-52}{-28} \qquad y = \frac{\overset{(33)-(25)}{\begin{vmatrix} 3 & 5 \\ 5 & 11 \end{vmatrix}}}{-28} = \frac{8}{-28}$$

$$\boxed{\left(\tfrac{13}{7}, \tfrac{-2}{7}\right)}$$

Gabriel Cramer was a Swiss mathematician of the 18th century. He was a full professor at the University of Geneva at the age of 20. Even though this rule for solving systems bears his name, Cramer was not its originator. It was first published in 1748 by British mathematician Colin Maclaurin (two years before Cramer published).

Determinants & Cramer's Rule

PROBLEM SET 2.2

Find the value of each determinant:

1. $\begin{vmatrix} 7 & 8 \\ -8 & 0 \end{vmatrix}$

2. $\begin{vmatrix} 5 & 5 \\ 5 & 5 \end{vmatrix}$

3. $\begin{vmatrix} -6 & -2 \\ 2 & 6 \end{vmatrix}$

4. $\begin{vmatrix} 8 & -1 \\ 13 & 0 \end{vmatrix}$

5. $\begin{vmatrix} 24 & 6 \\ -13 & -4 \end{vmatrix}$

6. $\begin{vmatrix} 18 & -5 \\ -9 & 11 \end{vmatrix}$

7. $\begin{vmatrix} -13 & -11 \\ -17 & -12 \end{vmatrix}$

8. $\begin{vmatrix} -6 & 7 \\ -9 & 10 \end{vmatrix}$

Solve each system using Cramer's Rule:

9. $5x + 4y = -1$
   $2x - y = 10$

10. $3a + 8 = -b$
    $4a - 2b = -14$

11. $2x - 3y = 19$
    $6x + 3 = -6y$

12. $6a + 7 = 5b$
    $2a - 3b = 7$

<u>Review Problem</u>

Solve the following third order system:

13. $2x - y + 4z = 7$
    $x - 3y + z = -2$
    $3x - 2y + 2z = -2$

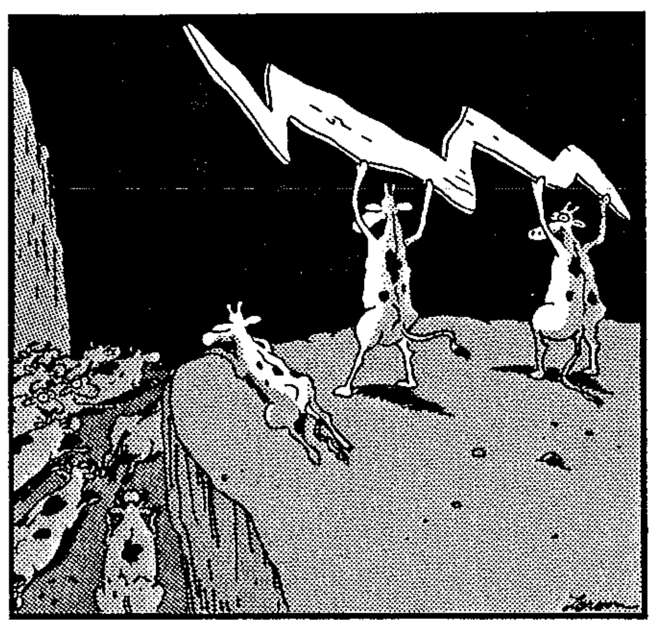
<u>Remembering Algebra I</u>

Classify each system as independent or dependent, consistent or inconsistent:

14. Two lines with different Slopes

15. Two parallel lines

16. Two equations that represent the same line



Calf delinquents

Expansion of Minors & Diagonals

DEMONSTRATION 2.3

Find the value of a
third order determinant:

<u>Expansion of Minors</u>

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$(a)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b)\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

<u>Diagonals</u>

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right|\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$aei + bfg + cdh - gec - hfa - idb$$

① $$\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$$

$$(2)\begin{vmatrix} 5 & 7 \\ 2 & 8 \end{vmatrix} - (3)\begin{vmatrix} 6 & 7 \\ 1 & 8 \end{vmatrix} + (4)\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix}$$

$$(40)-(14) \qquad (48)-(7) \qquad (12)-(5)$$

$$(2)(26) - (3)(41) + (4)(7)$$

$$52 - 123 + 28 = -43$$

③ $$\left|\begin{matrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{matrix}\right|\begin{matrix} 2 & 3 \\ 6 & 5 \\ 1 & 2 \end{matrix}$$

$$(80)+(21)+(48)-(20)-(28)-(144)$$

$$(80)+(21)+(48)+(-20)+(-28)+(-144)$$

$$-43$$

② $$\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & 1 & 2 \end{vmatrix}$$

$$(-1)\begin{vmatrix} -2 & -5 \\ 1 & 2 \end{vmatrix} - (4)\begin{vmatrix} 3 & -5 \\ -3 & 2 \end{vmatrix} + (0)\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}$$

$$(-4)-(-5) \qquad (6)-(15) \qquad (3)-(6)$$

$$(-1)(1) - (4)(-9) + (0)(-3)$$

$$(-1) - (-36) + (0) = 35$$

④ $$\left|\begin{matrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & 1 & 2 \end{matrix}\right|\begin{matrix} -1 & 4 \\ 3 & -2 \\ -3 & 1 \end{matrix}$$

$$(4)+(60)+(0)-(0)-(5)-(24)$$

$$(4)+(60)+(0)+(0)+(-5)+(-24)$$

$$35$$

Expansion of Minors & Diagonals

PROBLEM SET 2.3

Find the value of each determinant:

Expansion of minors

(1) $\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ -1 & 2 & 3 \end{vmatrix}$ (2) $\begin{vmatrix} -1 & 1 & 2 \\ 2 & 1 & 0 \\ 3 & 6 & -2 \end{vmatrix}$

(3) $\begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$ (4) $\begin{vmatrix} 3 & -1 & 2 \\ 0 & 4 & 1 \\ 5 & -2 & -3 \end{vmatrix}$

Diagonals

(5) $\begin{vmatrix} 1 & 2 & -3 \\ 3 & -5 & -1 \\ 4 & 4 & 1 \end{vmatrix}$ (6) $\begin{vmatrix} 3 & 2 & 5 \\ -1 & 1 & 1 \\ 4 & 3 & 3 \end{vmatrix}$

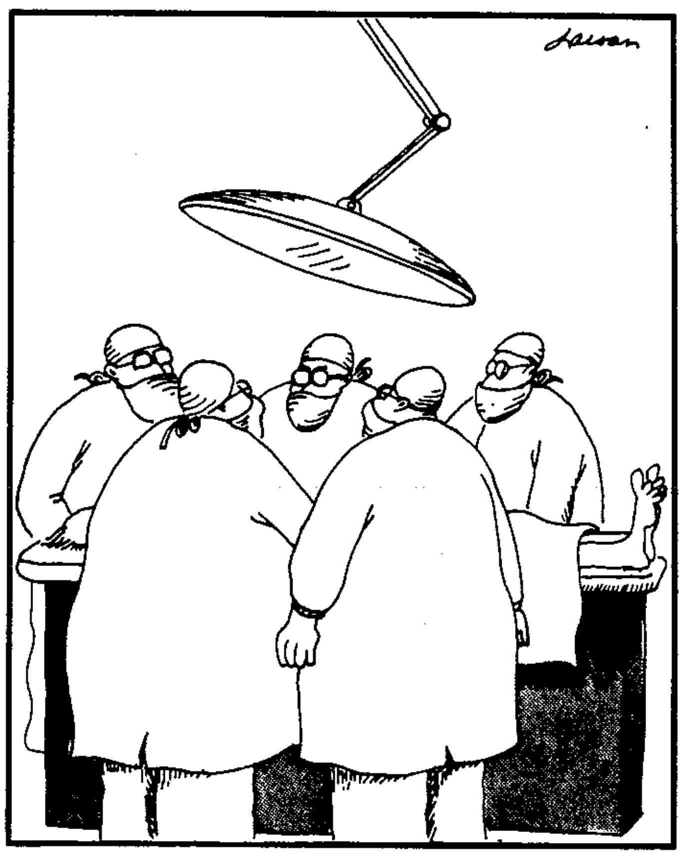
(7) $\begin{vmatrix} 8 & 3 & -2 \\ 4 & 1 & 1 \\ -2 & -5 & 6 \end{vmatrix}$ (8) $\begin{vmatrix} 3 & -5 & 2 \\ 4 & -3 & 4 \\ 2 & -1 & 5 \end{vmatrix}$

<u>Review Problems</u>

Use Cramer's Rule to solve the system

(9) $2x-3y=0$
$6x+5y=7$

(10) $3x+4y=6$
$2x+5y=11$



"Okay, Williams, we'll vote . . . how many here say the heart has four chambers?"

Cramer's Rule: Third Order

DEMONSTRATION 2.4

Solve a third order system using expansion of minors:

① $4x-3y+z=-1$
$2x+9y+5z=2$
$2x-6y-3z=0$

$$\begin{vmatrix} 4 & -3 & 1 \\ 2 & 9 & 5 \\ 2 & -6 & -3 \end{vmatrix}$$

$$(4)\begin{vmatrix} 9 & 5 \\ -6 & -3 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 5 \\ 2 & -3 \end{vmatrix} + (1)\begin{vmatrix} 2 & 9 \\ 2 & -6 \end{vmatrix}$$

$(-27)-(-30)$ $\quad (-6)-(10)$ $\quad (-12)-(18)$

$(4)(3)-(-3)(-16)+(1)(-30)=-66$

$$x=\frac{\begin{vmatrix} -1 & -3 & 1 \\ 2 & 9 & 5 \\ 0 & -6 & -3 \end{vmatrix}}{-66} \qquad y=\frac{\begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 5 \\ 2 & 0 & -3 \end{vmatrix}}{-66}$$

$$z=\frac{\begin{vmatrix} 4 & -3 & -1 \\ 2 & 9 & 2 \\ 2 & -6 & 0 \end{vmatrix}}{-66}$$

$x = -33/-66 = 1/2$
$y = -44/-66 = 2/3$
$z = 66/-66 = -1$

$(1/2, 2/3, -1)$

Solve a third order system using diagonals

② $4x-3y+z=-1$
$2x+9y+5z=2$
$2x-6y-3z=0$

$$\left|\begin{array}{ccc|cc} 4 & -3 & 1 & 4 & -3 \\ 2 & 9 & 5 & 2 & 9 \\ 2 & -6 & -3 & 2 & -6 \end{array}\right|$$

$(-108)+(-30)+(-12)-(18)-(-120)-(18)$

Denominator $=-66$

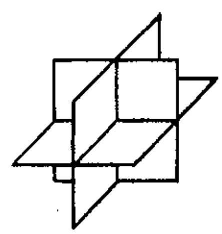
$$x=\frac{\left|\begin{array}{ccc|cc} -1 & -3 & 1 & -1 & -3 \\ 2 & 9 & 5 & 2 & 9 \\ 0 & -6 & -3 & 0 & -6 \end{array}\right|}{-66} \qquad y=\frac{\left|\begin{array}{ccc|cc} 4 & -1 & 1 & 4 & -1 \\ 2 & 2 & 5 & 2 & 2 \\ 2 & 0 & -3 & 2 & 0 \end{array}\right|}{-66}$$

$$z=\frac{\left|\begin{array}{ccc|cc} 4 & -3 & -1 & 4 & -3 \\ 2 & 9 & 2 & 2 & 9 \\ 2 & -6 & 0 & 2 & -6 \end{array}\right|}{-66}$$

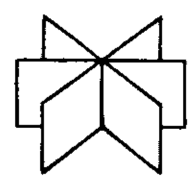
$x = -33/-66 = 1/2$
$y = -44/-66 = 2/3$
$z = 66/-66 = -1$

$(1/2, 2/3, -1)$

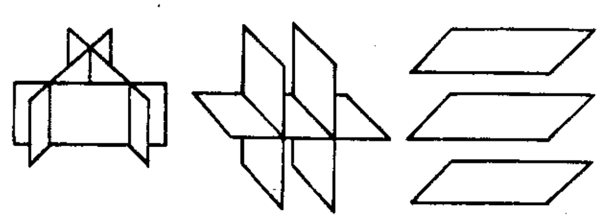
Note: A third order system does not have a unique solution if the denominator is "0." Therefore, always compute the denominator first when using Cramer's Rule.



one solution



infinite solutions



Systems with no solutions

Cramer's Rule: Third Order

PROBLEM SET 2.4

Use expansion of minors to solve each system:

① $2x - y + z = -2$
$x + 2y + 6z = 3$
$3x - y + 2z = -1$

② $3x - y + 2z = 11$
$6x - 3y + z = -1$
$-3x - 2y + 2z = 11$

③ $2x - y + 3z = 4$
$3x + 2y + z = 7$
$x + 3y - 2z = 3$

Use diagonals to solve each system:

④ $a + 2b - 3c = -13$
$2a - b + 3c = 23$
$3a + b - 3c = -8$

⑤ $5x - y + 2z = 5$
$2x - 3y + 5z = 1$
$3x + 2y - 3z = 4$

⑥ $x + 4y + 3z = 10$
$2x - 2y + z = 15$
$x + 2y - 3z = -1$

Review Problems

Solve using substitution:

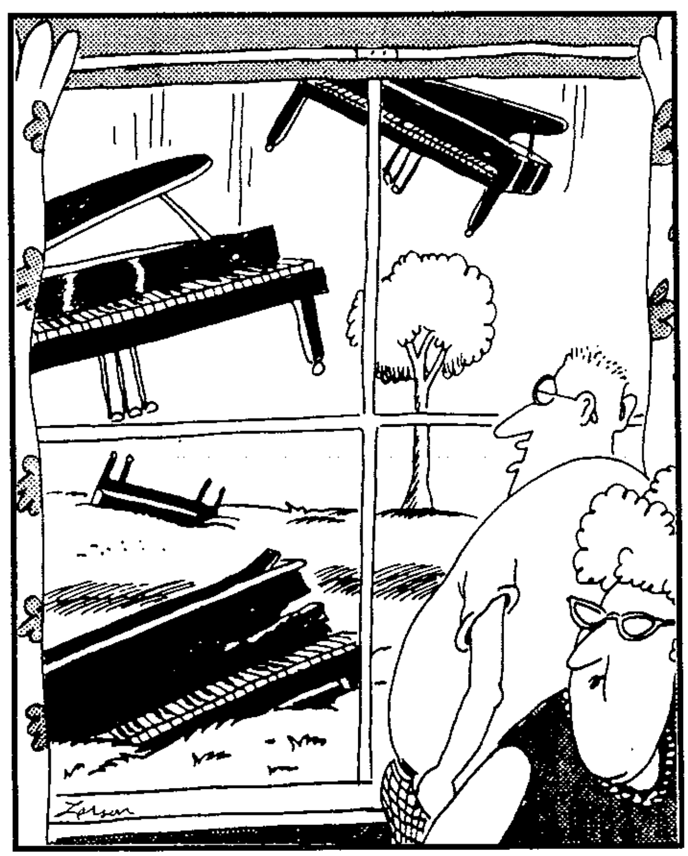
⑦ $x + 2y = 5$
$x - y = 6$

Solve using elimination:

⑧ $3x - 5y = -13$
$4x + 3y = 2$

Solve using Cramer's Rule:

⑨ $7x - 8y = 11$
$9x - 2y = 3$



"My word! I'd hate to be caught outside on a day like this!"

Systems

UNIT 2 REVIEW & PRACTICE

Solve using substitution:

① $$\begin{aligned} x - 3y &= -3 \\ 4x + 9y &= 2 \end{aligned}$$

Solve using elimination:

② $$\begin{aligned} 2x + 3y &= 5 \\ -3x + 6y &= 12 \end{aligned}$$

Solve using elimination and substitution:

③ $$\begin{aligned} 4x + 3y + z &= -10 \\ x - 12y + 2z &= -5 \\ x + 18y + z &= 4 \end{aligned}$$

Determine the value:

④ $\begin{vmatrix} -3 & 7 \\ 4 & 9 \end{vmatrix}$ ⑤ $\begin{vmatrix} -2 & 0 \\ 6 & 5 \end{vmatrix}$

Solve using Cramer's Rule:

⑥ $$\begin{aligned} 3x + 2y &= 40 \\ x - 7y &= -2 \end{aligned}$$

Determine the value using expansion of minors:

⑦ $$\begin{vmatrix} 4 & -5 & 3 \\ -3 & 2 & 0 \\ 2 & -1 & 4 \end{vmatrix}$$

Determine the value using diagonals:

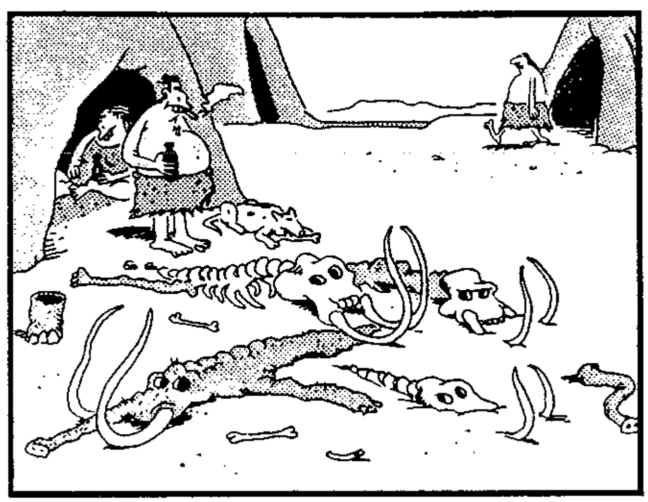
⑧ $$\begin{vmatrix} 6 & 3 & 1 \\ -1 & 1 & -4 \\ -2 & 4 & 2 \end{vmatrix}$$

Use expansion of minors to determine if a unique solution exists:

⑨ $$\begin{aligned} 2a - b + 3c &= 5 \\ 3a + 2b - 5c &= 7 \\ a - 4b + 11c &= 3 \end{aligned}$$

Solve for "x" using diagonals:

⑩ $$\begin{aligned} 3x + 4y + z &= 10 \\ 6x - 2y - z &= 6 \\ 3x + 6y - 2z &= 2 \end{aligned}$$



Of course, prehistoric neighborhoods always had that one family whose front yard was strewn with old mammoth remains.

UNIT 3

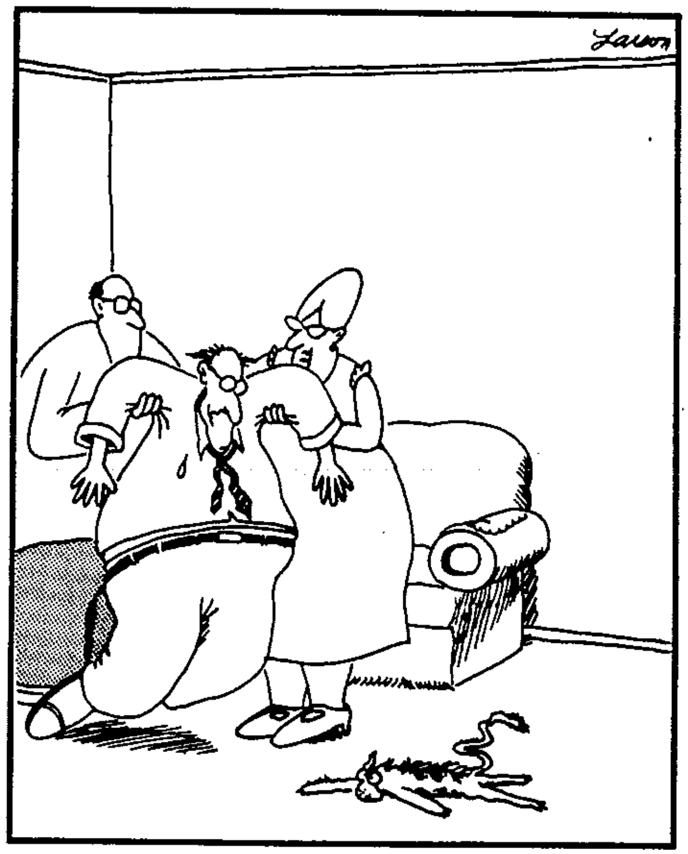
Problem Solving

Problem Types:

Rate, Time, Distance
Three Digit Number Problems
Mixture Problems
Sales Commission
Sales Tax
Rate of Discount
Investment Interest
Clock Problems (Rate)

Emphasis:

Problems Involving Systems of Equations



"Well, I guess both Warren and the cat are okay . . . But thank goodness for the Heimlich maneuver!"

Problem Solving

DEMONSTRATION 3.1

Problem Set "A" includes basic problems involving rate-time-distance, mixture, sales and investment, and digit problems involving systems.

① Mr. Davis made the trip to the lake in 3 hours. The return trip took 3½ hours. He drove 5 mph faster on his way out to the lake. How many miles was the round trip?

	R	× T	= D
to the lake	$(r+5)$	$\times 3$	$= (3r+15)$
return trip	r	$\times 3\frac{1}{2}$	$= 3\frac{1}{2}r$

$3r+15=3.5r$ $\quad 2(3.5r)$

$-.5r=-15$

$r=30$ $\quad \boxed{210 \text{ miles}}$

② The sum of the digits of a three digit number is 10. When the digits are reversed, the new number is 99 less than the original. The hundreds digit equals the sum of the tens and units digits. Find the number.

a) $h+t+u=10$

b) $100u+10t+h=(100h+10t+u)-99$

$99u-99h=-99$

$u-h=-1$

c) $h=t+u$

a) $h+t+u=10$

c) $h-t-u=0$

$2h = 10$

$h=5$

b) $u-h=-1$

$u-(5)=-1$

$u=4$

c) $h=t+u$

$(5)=t+(4)$

$t=1$

$\boxed{514}$

Remembering Algebra I

③ How many liters of a 70% salt solution should be mixed with 60 liters of a 20% solution to produce a 40% solution?

$.20(60)+.70(n)=.40(60+n)$

$12+.7n=24+.4n$

$.3n=12$ $\quad n=\boxed{40 \text{ liters}}$

④ Sharon earns a weekly salary of \$150 plus a 9% commission on sales. What are her sales if she brings home \$1098 over a three week period?

(continued)

Problem Solving

DEMONSTRATION 3.1

$3(150) + .09(n) = 1098$
$.09n = 648$
$n = 7200$ $\boxed{\$7200}$

⑤ Bud invested \$7000, part at 8% and part at 10%. His total interest for the year was \$650. How much was invested at 8%?

$.08(n) + .10(7000 - n) = 650$
$.08n + 700 - .1n = 650$
$-.02n = -50$
$n = 2500$ $\boxed{\$2500}$

⑥ Michael paid \$23.85 for his purchase after a 6% sales tax was added. Find the original price and the amount of tax.

$$\begin{array}{lcc} \text{aft. tax} & \dfrac{23.85}{n} = \dfrac{106}{100} \\ \text{org. pr} & \end{array}$$

$n = 22.5$ $\boxed{\$22.50, \$1.35}$

Problem Set "B" takes the same basic problem types and makes them slightly more complex – requiring creative problem solving.

⑦ Maria rode her bike 6 miles into town with the wind at her back in 18 minutes. Against the wind on her way back, she rode only 4 miles in 30 minutes. Find the speed of the wind.

	R	·	T	=	D
to town	$(r+w)$	·	$3/10$	=	6
back home	$(r-w)$	·	$1/2$	=	4

$(3/10\,r + 3/10\,w = 6) \cdot 10$ $\quad 3r + 3w = 60$
$(1/2\,r - 1/2\,w = 4) \cdot 6$ $\quad 3r - 3w = 24$
$6r = 84$
$r = 14$
$w = 6$

$\boxed{6 \text{ mph}}$

⑧ Stacy bought a new clock on sale. She paid \$11.55. The clock was on sale for 25% off. After the discount, she also had a coupon for an additional \$2.50 off. After the coupon, she had to pay a 5% sales tax. Find the original price.

aft. tax / purch pr: $\dfrac{11.55}{n} = \dfrac{105}{100}$ $\quad 105n = 1155$, $n = 11$

sale pr / org. pr: $\dfrac{13.50}{n} = \dfrac{75}{100}$ $\quad 75n = 1350$, $n = 18$

(continued)

Problem Solving

DEMONSTRATION 3.1

\$11.00 before tax
\$ 2.50 add back coupon
\$13.50

$\boxed{\text{\$18 original price}}$

	R	·	T	=	D
minute hand	60	·	t	=	60t
hour hand	5	·	t	=	5t

The hour hand has a 40 minute head start at 8:00

Problem Set "C" focuses exclusively on rate-time-dist. problems that require more creative application.

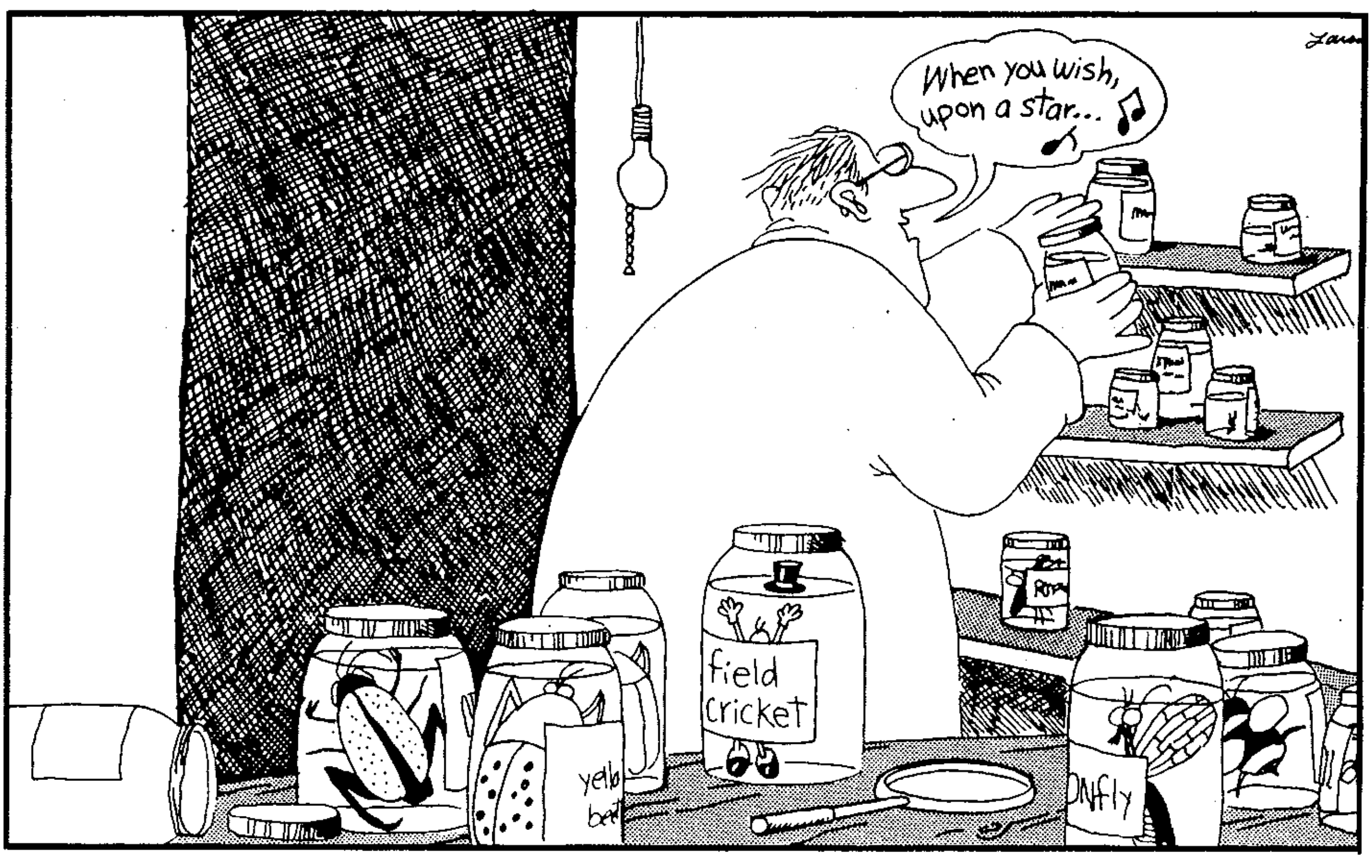
$$60t = 5t + 40$$
$$55t = 40$$
$$t = 8/11 \text{ hours}$$

⑨ At what time between 8:00 and 9:00 will the clock hands coincide (to the nearest second)?

$(8/11) \cdot (60 \text{ min}) = 43\ 7/11 \text{ min.}$

$(7/11) \cdot (60 \text{ sec}) = 38\ 2/11 \text{ sec.}$

$\boxed{8:43:38}$



Problem Solving

PROBLEM SET 3.1

Problem Set "A"

① Two hours after a truck leaves Phoenix traveling at 45mph, a car leaves to overtake the truck. If it takes the car 10 hours to catch the truck, what is the speed of the car?

② At 2:00 PM two cars start toward each other from towns 240 miles apart. If the rate of one car is 10 mph faster than the other, what are their rates if they meet at 5:00?

③ If Jim increases his cycling speed by 4mph, he can cover a distance in 2 hours that usually takes him 3 hours. What is the new speed?

④ Vera is driving to Lake Stratton, a distance of 662 miles. If she drives 55 mph for 5 hours, at what speed must she travel to complete the trip in 14 hours?

⑤ The sum of the digits of a three digit number is 15. The hundreds digit exceeds the sum of the other two by 1. When the digits are reversed, the new number is 495 less than the original. Find the original number.

⑥ The sum of the digits of a three digit number is 17. The tens digit is three times the hundreds digit. The sum of the hundreds digit and the tens digit is 1 less than the units digit. Find the number.

⑦ The perimeter of a triangle is 45 cm. The two shorter sides differ by 2 cm. The longest side is 7cm less than the sum of the other two. Find the length of each side.

⑧ The largest angle of a triangle is 15 degrees greater than the smallest. The sum of the two larger angles exceeds twice the smallest by 24 degrees. Find the measure of the middle angle.

⑨ How much pure copper should be mixed with 20 kg of a 40% alloy and 10 kg of a 50% alloy to produce

Problem Solving

PROBLEM SET 3.1

a 66% alloy?

(10) How many liters of a 70% salt solution should be added to 30 liters of a 90% solution to produce an 82% solution?

(11) Erin earns $240 per week plus 8% commission on sales. If she takes home $680 in one week, what were her sales?

(12) Janice paid $23.52 for her purchase after a 5% tax was added on. How much tax did she pay?

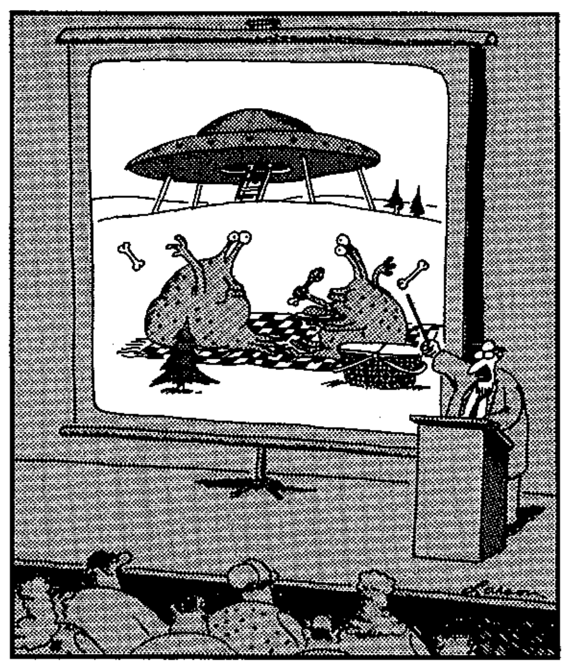
(13) Brian invested $8000, part at 8% and part at 6%. After one year, his 8% investment earned $40 less than twice as much as the 6% investment. How much did he invest at each rate?

(14) Lynn paid $27.88 for some fabric. If the fabric was on sale at 15% off, how much did she save on her purchase?

Problem Set "B"

(15) With the wind behind him, an ice skater crossed a frozen river 1 mile wide in 3 minutes. On his return, he skated straight into the wind and it took 15 minutes. Find the speed of the wind.

(16) A crew rows 3 miles upstream in 20 minutes and 3 miles downstream in 12 minutes. Find their rate in still water.



Professor Ferrington and his controversial theory that dinosaurs were actually the discarded "chicken" bones of giant, alien picnickers.

Problem Solving

PROBLEM SET 3.1

(17) Kate rode her bike at 15 mph. After she had a flat tire, she had to walk at 4 mph to finish her trip. If she covered 53 miles in 5 hours, how far did she walk?

(18) The sum of the digits of a three digit number is 14. The tens digit exceeds twice the hundreds digit by 1. If the units digit is increased by 3, the units digit and the hundreds digit are the same. Find the number.

(19) In a three digit number, twice the hundreds digit exceeds the sum of the tens and units digit by 3. When the digits are reversed, the new number is 198 greater than the original. The sum of the digits is 12. Find the original number.

(20) How much water must be removed from 32 liters of a 25% acid solution to raise the acid concentration to 40%?

(21) Takashi earns a regular weekly salary plus $7\frac{1}{2}$% commission on sales. During a recent 4-week period, his sales were exactly 100 times as much as his weekly salary. If he took home $1380 over the 4 weeks, determine his weekly salary.

(22) Teri purchased a book on sale for 10% off. After deducting the discount, the sales clerk added a 5% sales tax. If Teri paid $11.34, what was the original price of the book?

(23) Jennifer invested $4000, part at 8% and part at 6% annual interest. She earned $100 more from the 6% investment. How much interest did Jennifer earn for the year - and - what rate of interest does this represent on the entire $4000 investment?

Problem Set "C"

(24) At what time between 5:00 and 6:00 do the hands of a clock coincide (to the nearest second)?

Problem Solving

PROBLEM SET 3.1

(25) At what time between 3:00 and 4:00 do the hands of a clock coincide (to the nearest second)?

(26) Conrad can jog to Chris's house in 10 minutes. Chris can ride his bike to Conrad's house in 6 minutes. If they start from their houses one mile apart at the same time, in how many minutes will they meet?

(27) A boy leaves on a bicycle trip at the rate of 8 mph. One hour later, his father realizes that the boy forgot some camping gear, and he sets out by car to catch him. If the father catches up in 15 minutes, how fast is he driving?

(28) Lars jogs to work at 8 mph going uphill. He jogs home at 12 mph in 15 minutes less time. How far does he live from work?

(29) A man has $3\frac{1}{4}$ hours to give some friends a tour of the surrounding countryside. How far from the house can the tour extend if he drives at 25 mph out and 40 mph back?

(30) With the wind, a plane travels 1200 miles in 3 hours. The same trip takes 5 hours against the wind. How far can the plane travel with a normal tailwind for 2 hours followed by no wind at all for another 2 hours? (one distance)

"Well, you've overslept and missed your vine again."

UNIT 17

Sequence & Series

Appliance healers

Binomial Theorem

DEMONSTRATION 17.1

The coefficients of a basic binomial expansion form a pattern called Pascal's Triangle. This pattern was popularized by Blaise Pascal (1623–1662) who was a French mathematician.

PASCAL'S TRIANGLE

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Each row begins and ends with 1.

Each term is the sum of the two terms to the left and right in the row directly above.

Each row is symmetrical.

Without constructing the entire triangle, it is possible to determine the values in a specific row.

① List the terms from the 11th row of Pascal's Triangle.

There will be 11 terms:

1 $\quad 10\left(\frac{9}{2}\right) \quad 45\left(\frac{8}{3}\right) \quad 120\left(\frac{7}{4}\right) \quad 210\left(\frac{6}{5}\right)$ 252 210 120 45 10 1

② List the terms from the 12th row of Pascal's Triangle.

There will be 12 terms:

1 $\quad 11\left(\frac{10}{2}\right) \quad 55\left(\frac{9}{3}\right) \quad 165\left(\frac{8}{4}\right) \quad 330\left(\frac{7}{5}\right) \quad 462\left(\frac{6}{6}\right)$ 462 330 165 55 11 1

The Binomial Theorem uses this pattern for coefficients in a binomial expansion. This pattern is combined with a descending pattern of first term exponents and an ascending pattern of the second term exponents.

If the binomial is a sum, all signs are "+" in the expansion. If the binomial is a difference, the signs will alternate.

Binomial Theorem

DEMONSTRATION 17.1

The coefficients of the expansion form the pattern of the row in Pascal's Triangle one greater than the exponent.

③ Expand $(a+b)^5 \to 1 \quad 5(\frac{4}{2}) \quad 10(\frac{3}{3}) \quad 10 \quad 5 \quad 1$

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

④ Expand $(x-y)^8 \to 1 \quad 8(\frac{7}{2}) \quad 28(\frac{6}{3}) \quad 56(\frac{5}{4}) \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$

$x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$

⑤ Expand $(2n-m)^4 \to 1 \quad 4(\frac{3}{2}) \quad 6 \quad 4 \quad 1$

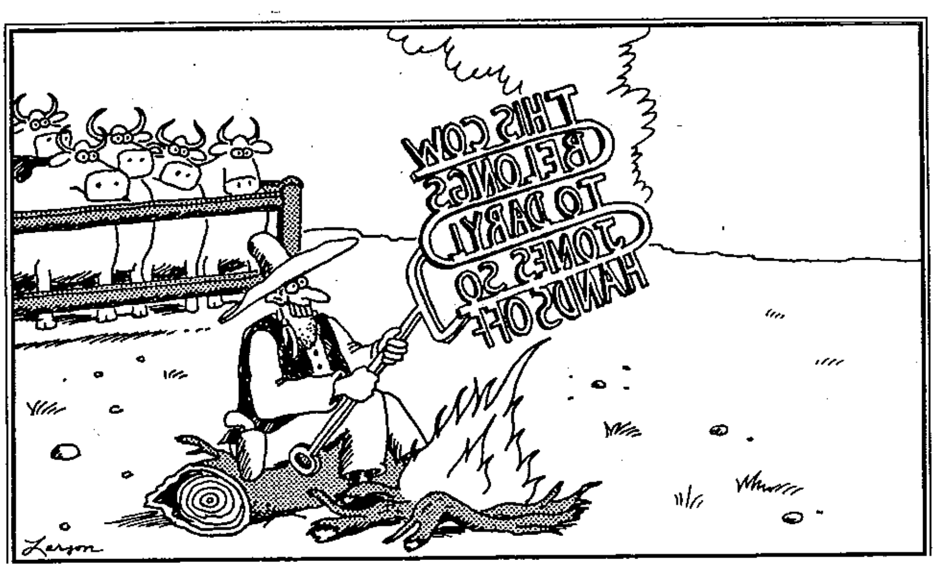
$(2n)^4 - 4(2n)^3m + 6(2n)^2m^2 - 4(2n)m^3 + m^4$

$16n^4 - 32n^3m + 24n^2m^2 - 8nm^3 + m^4$

⑥ Expand $(3x+2y)^7 \to 1 \quad 7(\frac{6}{2}) \quad 21(\frac{5}{3}) \quad 35(\frac{4}{4}) \quad 35 \quad 21 \quad 7 \quad 1$

$(3x)^7 + 7(3x)^6(2y) + 21(3x)^5(2y)^2 + 35(3x)^4(2y)^3 + 35(3x)^3(2y)^4 + 21(3x)^2(2y)^5$
$+ 7(3x)(2y)^6 + (2y)^7$

$2187x^7 + 10{,}206x^6y + 20{,}412x^5y^2 + 22{,}680x^4y^3 + 15{,}120x^3y^4 +$
$6048x^2y^5 + 1344xy^6 + 128y^7$



⑦ Review:

$x^5 + y^5 =$

$(x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

Factoring like odd powers

Binomial Theorem

PROBLEM SET 17.1

List the terms from the indicated row of Pascal's Triangle:

① 6th row (exponent 5)

② 5th row (exponent 4)

③ 9th row (exponent 8)

④ 14th row (exponent 13)

Expand each binomial:

⑤ $(x+m)^4$

⑥ $(r+s)^6$

⑦ $(b-z)^5$

⑧ $(r-m)^8$

⑨ $(3r+y)^4$

⑩ $(2b+x)^6$

⑪ $(3x-2y)^5$

⑫ $(2m-3)^6$

⑬ $(2n+3m)^8$

⑭ $(x-4y)^9$

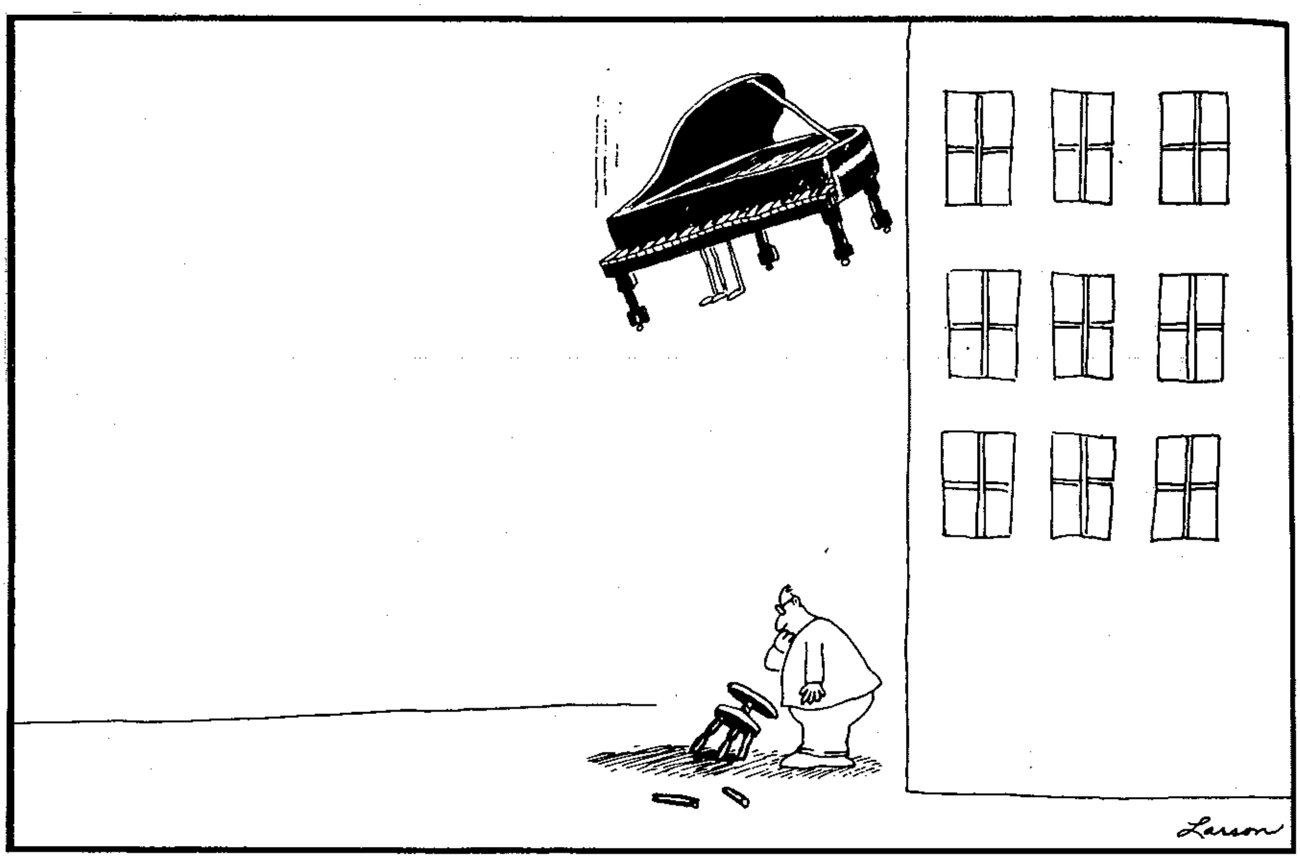
Factor:

⑮ a^5-b^5

⑯ $32x^5+y^{10}$

⑰ $128a^{14}-b^{21}$

⑱ $243x^{15}-1$



Sigma Notation & Arithmetic Series

DEMONSTRATION 17.2

An arithmetic series is the sum of a sequence of numbers with a common difference.

Arithmetic Series

Example: $3+7+11+15+19$

$n=5$ number of terms
$d=4$ common difference
$a_1=3$ first term

Sum: Arithmetic Series

Formula:

$$S = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

Sigma Notation

$$\sum_{x=3}^{7} (3x-2)$$ Summation from 3 to 7 of $(3x-2)$

$n=5$ number of terms $(7-3)+1=5$

$d=3$ common difference coefficient of x

$a_1=7$ 1st term $3(3)-2=7$

Find the sum and state the first three terms:

① $$\sum_{x=2}^{9} (4x+3)$$

$n=8$
$d=4$
$a_1=11$

$S = \frac{n}{2}\left(2a_1 + (n-1)d\right)$

$S = \frac{8}{2}\left(2(11) + 7(4)\right)$

$S = 4\,(50) = \boxed{200}$ $\boxed{11+15+19}$

Solve

② Find the sum of the positive multiples of 9 less than 305.

$n = 33$ $(305 \div 9 = 33.\overline{8})$
$d = 9$
$a_1 = 9$

$S = \frac{33}{2}\left(2(9) + 32(9)\right)$

$S = \frac{33}{2}\,(306) = \boxed{5049}$

Sigma Notation & Arithmetic Series

PROBLEM SET 17.2

Indicate the sum for each series and write the first three terms:

① $\sum_{k=2}^{6} (3+2k)$

② $\sum_{m=3}^{9} (5-3m)$

③ $\sum_{x=2}^{10} (8-3x)$

④ $\sum_{n=1}^{30} (2n-1)$

⑤ $\sum_{n=1}^{40} (3n+2)$

⑥ $\sum_{x=10}^{50} (3x-1)$

⑦ $\sum_{m=21}^{75} (2m+5)$

Solve each problem:

⑧ Find the sum of the odd integers from 1 to 100.

⑨ Find the sum of the positive integers less than 100 and divisible by 6.

⑩ An auditorium has 21 seats in the first row. Each row has one more seat than the row in front of it. If there are 30 rows, what is the seating capacity of the auditorium?

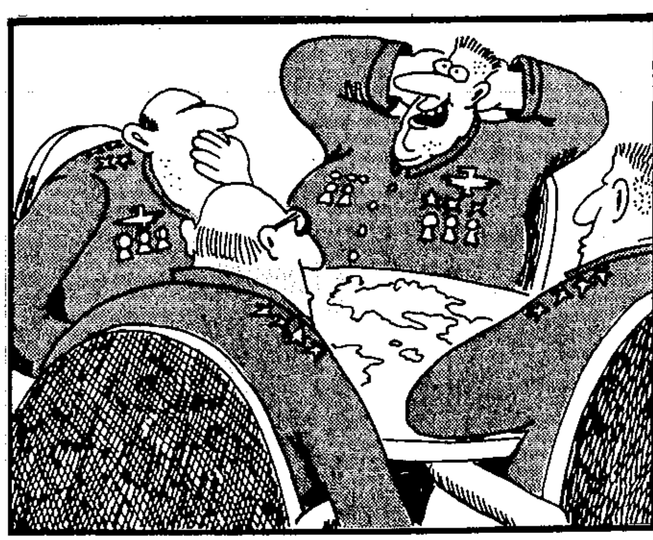
<u>Review</u>: Expanding/Factoring

⑪ $(m+3n)^7$

⑫ $(2x-5y)^6$

⑬ $256x^{14}-2y^{21}$

⑭ $1024a^5b^{10}+c^{15}$



"On the other hand, gentlemen, what if we gave a war and EVERYBODY came?"

Geometric Series

DEMONSTRATION 17.3

A geometric series is the sum of a sequence of numbers with a common ratio.

Geometric Series

Example: $2+(-6)+18+(-54)+162$

$n=5$ number of terms
$r=-3$ common ratio
$a_1=2$ first term

Sum: Geometric Series

Formula:

$$S = \frac{a_1 - a_1 r^n}{1-r}$$

Sigma Notation

$$\sum_{x=-1}^{5} 2(4)^{x-1}$$

Summation from -1 to 5 of $2(4)^{x-1}$

$n=7$ number of terms
$5-(-1)+1=7$

$r=4$ common ratio
base number (4)

$a_1=1/8$ first term
$2(4)^{-2}$

Find the sum and state the first three terms:

① $$\sum_{x=2}^{9} 2(-3)^x$$

$n=8$
$r=-3$
$a_1=18$

$$S = \frac{a_1 - a_1 r^n}{1-r} = \frac{18-18(-3)^8}{1-(-3)}$$

$\boxed{-29,520}$ $\boxed{18+(-54)+162}$

② $$\sum_{x=-1}^{8} 4(2)^{x-1}$$

$n=10$
$r=2$
$a_1=1$

$$S = \frac{a_1 - a_1 r^n}{1-r} = \frac{1-1(2)^{10}}{1-2}$$

$\boxed{1023}$ $\boxed{1+2+4}$

Geometric Series

PROBLEM SET 17.3

Determine if the series is arithmetic or geometric. Find the sum. List the first three terms:

① $\sum_{c=1}^{8} 3^c$

② $\sum_{t=-2}^{6} 3(2)^{t+3}$

③ $\sum_{x=-3}^{3} (2x+2)$

④ $\sum_{e=3}^{12} 3(-2)^e$

⑤ $\sum_{n=0}^{6} (24-9n)$

⑥ $\sum_{k=1}^{7} 2^{k-2}$

⑦ $\sum_{n=2}^{11} 4(-3)^{n-1}$

Review Problems

Expand:

⑧ $(x-y)^7$

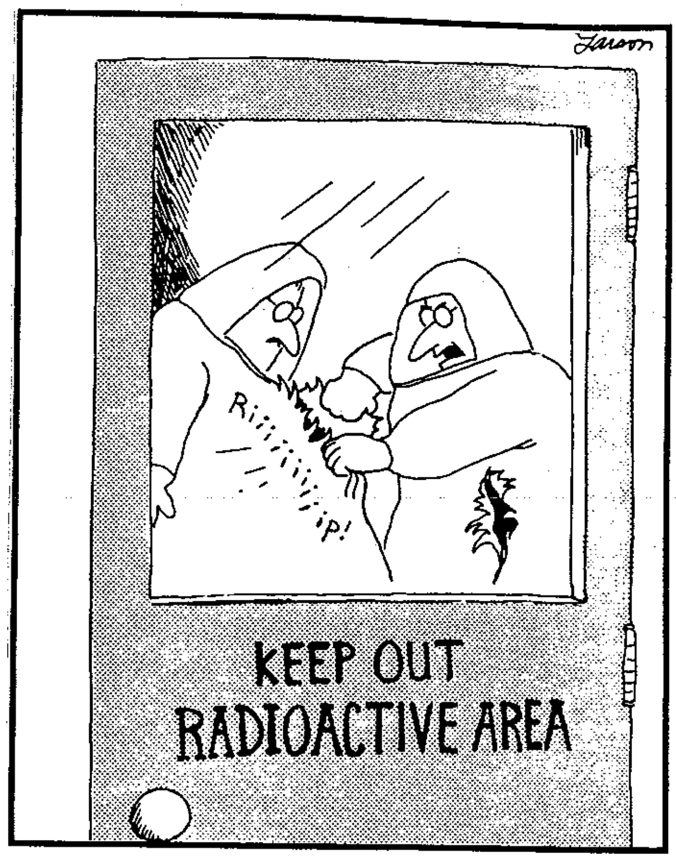
⑨ $(2x+3y)^5$

⑩ $(3n-4m)^6$

Factor:

⑪ $64x^5-2y^{10}$

⑫ $a^7b^7+128c^7$



"So, Foster! That's how you want it, huh? ... Then take THIS!"

Infinite Geometric Series

DEMONSTRATION 17.4

If a geometric series has an infinite number of terms, it is an infinite geometric series. If the common ratio is between -1 and 1, a value very close to the sum can be determined.

Infinite Geometric Series

Example: $4+1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$

$a_1 = 4$ first term

$r = 1/4$ Common ratio

Sum: Infinite Geometric Series

Formula:

$$S = \frac{a_1}{1-r} \text{ for } -1<r<1$$

① $\frac{4}{3}-\frac{2}{3}+\frac{1}{3}-\frac{1}{6}+\cdots$

$a_1 = 4/3$

$r = (\frac{-2}{3}) \div (\frac{4}{3}) = \frac{-1}{2}$

$S = \frac{a_1}{1-r} = \frac{4/3}{3/2} = \boxed{\frac{8}{9}}$

② $\frac{8}{3}-8+24-72+216-\cdots$

$a_1 = 8/3$

$r = (-8) \div (8/3) = -3$ $\boxed{\text{no sum}}$

The common ratio is not between -1 and 1

Arithmetic Series: Summation Notation

Example: $5+8+11+14+17$

$$\sum_{x=1}^{n} (dx+\square)$$

Geometric Series: Summation Notation

Example: $4+(-8)+16+(-32)$

$$\sum_{x=1}^{n} \square r^x$$

③ Example: $5+8+11+14+17$

$$\sum_{x=1}^{5} (3x+2)$$

$n=5$
$d=3$

④ Example: $4+(-8)+16+(-32)$

$$\sum_{x=1}^{4} -2(-2)^x$$

$n=4$
$r=-2$

Infinite Geometric Series

DEMONSTRATION 17.4

Determine the summation notation for each series:

⑤ $5+9+13+17+21$

$$\sum_{x=1}^{5} (4x+1) \qquad \begin{matrix} n=5 \\ d=4 \end{matrix}$$

⑥ $1-3+9-27+81-243$

$$\sum_{x=1}^{6} -\tfrac{1}{3}(-3)^x \qquad \begin{matrix} n=6 \\ r=-3 \end{matrix}$$

Solve:

⑦ The end of a swinging pendulum moves 20 cm on the first swing and 10/11 of the preceding distance on each successive swing. How far will it move in total?

$a_1 = 20$
$r = 10/11$

$$S = \frac{a_1}{1-r} = \frac{20}{1-10/11} = \frac{20}{1/11}$$

220 cm

⑧ A rubber ball that is dropped 30 feet bounces 2/5 of the height from which it fell on each bounce. How far will it travel?

$a_1 = 30 \quad r = .4$

downward

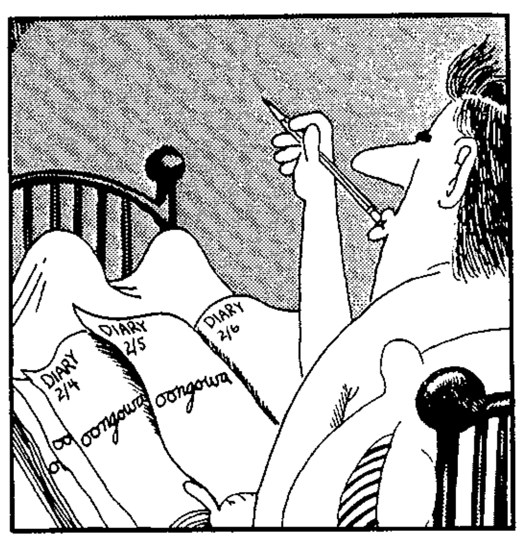
$$S = \frac{a_1}{1-r} = \frac{30}{.6} = 50$$

$a_1 = (30)(\tfrac{2}{5}) = 12 \quad r = .4$

upward

$$S = \frac{a_1}{1-r} = \frac{12}{.6} = 20$$

$50 + 20 = 70$ feet



Tarzan contemplates another entry.

Infinite Geometric Series

PROBLEM SET 17.4

Determine the sum (if possible):

① $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots$

② $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$

③ $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

④ $1 - 3 + 9 - 27 + \ldots$

Determine the summation notation for each series:

⑤ $2 + 6 + 10 + 14 + 18$

⑥ $14 + 8 + 2 + (-4) + (-10)$

⑦ $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4}$

⑧ $9 - 3 + 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}$

⑨ $\frac{1}{6} - \frac{1}{12} + \frac{1}{24} - \frac{1}{48} + \frac{1}{96}$

⑩ $40 + 10 + \frac{5}{2} + \frac{5}{8} + \frac{5}{32}$

Solve:

⑪ A hot-air balloon rises 80 feet in the first minute of flight. If in each succeeding minute the balloon rises only 90% as far as in the previous minute, what will be its maximum altitude?

⑫ A silicon ball dropped 12 feet rebounds 7/10 of the height from which it fell on each bounce. How far will it travel before coming to a rest?

Review

⑬ Expand $(2a - 3b)^8$

⑭ Determine the sum and first three terms:

$$\sum_{x=-2}^{11} (4x-3)$$

⑮ Determine the sum and first three terms:

$$\sum_{a=4}^{12} 2(4)^{a-5}$$

⑯ Factor $x^5y^{10} + 32z^{10}$

Sequence & Series

UNIT 17 REVIEW & PRACTICE

Expand each binomial:

① $(x+y)^7$

② $(3a+4b)^4$

③ $(n-3m)^5$

④ $(2x-3y)^6$

Indicate the sum for each series and the first 3 terms:

⑤ $\sum_{x=1}^{7}(4x-5)$

⑥ $\sum_{c=3}^{10}(8-5c)$

⑦ $\sum_{m=-2}^{9}(2)^m$

⑧ $\sum_{e=2}^{11}\frac{1}{2}(4)^{e-2}$

Find the sum if possible:

⑨ $3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

⑩ $8-6+\frac{9}{2}-\frac{27}{8}+\ldots$

Solve:

⑪ Find the sum of the even integers from 250 to 350.

⑫ A swinging pendulum moves 50 cm on its first swing and 9/10 of the preceding distance on each successive swing. How far will it move before stopping?

Determine the summation notation for each series:

⑬ $6-2-10-18-26$

⑭ $20+11+2-7-16-25$

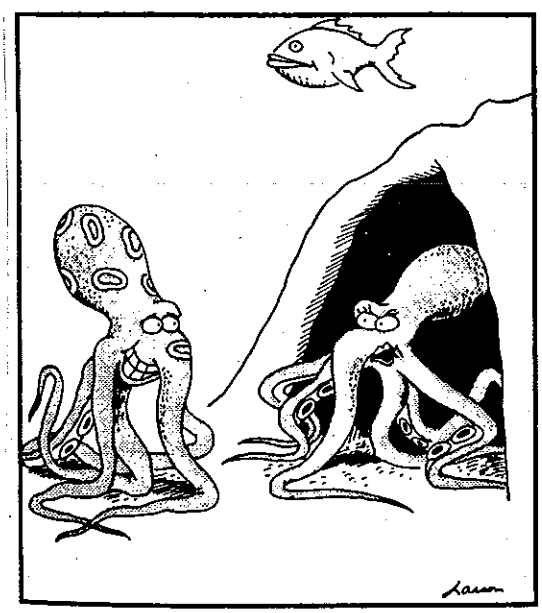
⑮ $3+9+27+81+243$

⑯ $10-5+\frac{5}{2}-\frac{5}{4}+\frac{5}{8}$

Factor completely:

⑰ $4x^5+972y^{10}$

⑱ $a^7b^{14}-128c^7$



"Oh yeah? . . . And I suppose you got those suction marks at the meeting, too!"

UNIT 18

Combinations & Permutations

18.1
Linear Permutations

18.2
Circular Permutations

18.3
Combinations

"When I got home, Harold's coat and hat were gone, his worries were on the doorstep, and Gladys Mitchell, my neighbor, says she saw him heading west on the sunny side of the street."

Linear Permutations

DEMONSTRATION 18.1

When events are unrelated, they are independent. If events are dependent, the number of choices is altered by the choice in previous events.

① How many seven-digit phone numbers begin with 457?

Indep: $1 \cdot 1 \cdot 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

The number of permutations of $\underline{n}$ objects taken $\underline{r}$ at a time is defined as: $P(n,r) = \dfrac{n!}{(n-r)!}$

③ Evaluate: $\dfrac{P(6,4)}{P(5,2)}$

$$\frac{\left(\frac{6!}{2!}\right)}{\left(\frac{5!}{3!}\right)} = \frac{6!\,3!}{2!\,5!} = 18$$

④ Evaluate: $P(5,5)$

$$\frac{5!}{0!} = \frac{5!}{1} = 5! = 120$$

note: $0! = 1$

⑤ How many ways can 7 different colored candles be arranged in a 3-candle candelabra?

$$P(7,3) = \frac{7!}{4!} = 210$$

② How many five-letter "words" can be formed using the letters: a, b, c, d, e, f?

Dep: $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$

The number of permutations of $\underline{n}$ objects when $\underline{p}$ are alike and $\underline{q}$ are alike: $\dfrac{n!}{p!\,q!}$

⑥ How many seven-letter patterns can be formed from "benzene"?

$$\frac{7!}{3!\,2!} = 420$$

⑦ How many ways can 3 algebra, 4 chemistry, and 5 history books be arranged by subject?

types	alg	chem	hist
$P(3,3)$	$P(3,3)$	$P(4,4)$	$P(5,5)$

$3!\,3!\,4!\,5! = 103{,}680$

Linear Permutations

PROBLEM SET 18.1

Identify each as dependent or independent and solve:

① The letters g, h, j, k, and m are used to form five-letter patterns. How many patterns can be formed if repetitions are allowed?

② A license plate must have two letters (not I or O) followed by three digits. The last digit cannot be zero. How many possible plates are there?

③ A store has 15 sofas, 12 lamps, and 10 tables at half price. How many different combinations of a sofa, a lamp, and a table can be purchased?

④ A car dealer offers a choice of 6 vinyl top colors, 18 body colors, and 7 upholstery colors. How many color combinations are there?

⑤ How many ways can six different books be placed on a shelf?

⑥ How many different 4-letter patterns can be formed from the letters a, e, i, o, r, s, and t if no letter occurs more than once?

Find the value of each:

⑦ $\dfrac{P(6,4)}{P(5,3)}$

⑧ $\dfrac{P(10,3)}{P(5,3)}$

⑨ $\dfrac{P(6,3)\cdot P(4,2)}{P(5,2)}$

⑩ $\dfrac{P(5,3)}{P(8,5)\cdot P(5,5)}$

Solve each problem:

⑪ Don has 5 pennies, 3 nickels, and 4 dimes. The coins of each denomination are indistinguishable. How many ways can he arrange the coins in a row?

⑫ Estelle has 8 quarters, 5 dimes, 3 nickels, and a penny. The coins of each denomination are indistinguishable. How many ways can she place the coins in a straight line?

⑬ In how many different

Linear Permutations

PROBLEM SET 18.1

orders can the numbers from 0 to 9 be arranged in a straight line?

(14) How many 6-digit numbers can be made using the digits from 833,284?

(15) There are 3 identical red flags and 5 identical white flags that are used to send signals. All 8 flags must be used. How many signals can be given?

(16) Five algebra and four geometry books are to be placed on a shelf. How many ways can they be arranged if all the algebra books are together?

And no one ever heard from the Anderson brothers again.

Circular Permutations

DEMONSTRATION 18.2

If <u>n</u> objects are arranged in a circle, there are $(n-1)!$ permutations.

If the <u>n</u> objects are arranged in a circle relative to a fixed point, the permutations are linear: $n!$ (Key chain with clasp)

If the <u>n</u> objects in a circle can be flipped over (Key chain with no clasp), there are only ½ the circular permutations: $\frac{(n-1)!}{2}$

① Five people are seated at a round table. How many seating arrangements are possible?

$(5-1)! = \boxed{24}$

circular, not reflective

② How many different ways can 5 charms be placed on a bracelet with no clasp?

$\frac{(5-1)!}{2} = \boxed{12}$

circular, reflective

③ How many different ways can 7 beads be arranged on a necklace that has a clasp?

$\frac{7!}{2} = \boxed{2520}$

linear, reflective

Circular Permutations

PROBLEM SET 18.2

State whether these arrangements are circular or linear, reflective or not reflective:

1. charms on a bracelet having no clasp
2. chairs around a table
3. paintings on a gallery wall
4. four people seated around a square table relative to each other
5. the batting order for a baseball team
6. beads on a necklace with a clasp

Solve:

7. How many ways can 6 keys be arranged on a key ring?
8. How many ways can 8 charms be arranged on a bracelet with no clasp?
9. How many ways can 5 people be seated at a round table relative to each other?
10. How many ways can 5 people be seated around a circular table if 2 of the people must be seated next to each other?
11. How many ways can 6 people be seated around a campfire?
12. How many ways can 4 men and 4 women be seated alternately at a round table?
13. How many ways can 6 people be seated at a round table relative to the door?
14. How many seating arrangements are possible for the President and 7 advisors, around a circular table if the President has a reserved seat?

<u>Review</u>

15. How many 5-digit numbers between 65,000 and 69,999 can be made with no repeats?
16. How many ways can 4 nickels and 5 dimes be given to 9 children if each receives one coin?

Combinations

DEMONSTRATION 18.3

The main difference between a permutation and a combination is whether "order" is considered. A permutation takes order into consideration. A combination does not.

A combination of n things taken r at a time is defined as: $C(n,r) = \frac{n!}{(n-r)!\,r!}$

① From a group of 6 men and 4 women, how many groups of 2 men and 3 women can be formed?

$$C(6,2) \cdot C(4,3) = \frac{6!}{4!2!} \cdot \frac{4!}{1!3!} = 60$$

② In an urn, there are 17 numbered discs. 8 are red, 5 are white, and 4 are blue. How many ways can 2 red, 1 white, and 2 blue discs be chosen?

$$C(8,2) \cdot C(5,1) \cdot C(4,2)$$

$$\frac{8!}{6!2!} \cdot \frac{5!}{4!1!} \cdot \frac{4!}{2!2!} = 840$$

③ Find the total number of diagonals in a regular decagon.

note: the sides are not considered to be diagonals.

$C(10,2) - 10$ (subtract the sides)

$$\frac{10!}{8!2!} - 10 = 35$$

④ From a deck of 52 cards, how many ways can 5 cards be drawn so that 3 are 1 suit and 2 are another?

$$C(13,3) \cdot C(13,2) \cdot P(4,2)$$

$$\frac{13!}{10!3!} \cdot \frac{13!}{11!2!} \cdot \frac{4!}{2!}$$

267,696

Combinations

PROBLEM SET 18.3

Evaluate:

① $C(8,5) \cdot C(7,3)$

② $C(24,21)$

Solve each:

③ From a list of 12 books, how many groups of 5 books can be selected?

④ How many baseball teams of 9 members can be formed from 14 players?

⑤ Suppose there are 9 points on a circle. How many quadrilaterals can be formed using the points as vertices?

⑥ There are 85 telephones at Kennedy High School. How many 2-way connections can be made among the school telephones?

⑦ How many different groups of 25 people can be formed from 27 people?

⑧ Suppose there are 8 points in a plane, no 3 of which are collinear. How many distinct triangles could be formed with these points as vertices?

⑨ From a deck of 52 playing cards, how many different 5-card hands have 5 cards of the same suit?

⑩ From a deck of 52 playing cards, how many different 4-card hands can have each card from a different suit?

A bag contains indistinguishable marbles. There are 4 red, 6 white, and 9 blue marbles. How many ways can 5 marbles be selected to meet the following conditions?

⑪ All the marbles are blue

⑫ All the marbles are red

⑬ Two are red, two are white, and one is blue

* ⑭ Two are one color and three are another color

* This is a long problem requiring extensive set-up.

(Continued)

Combinations

PROBLEM SET 18.3

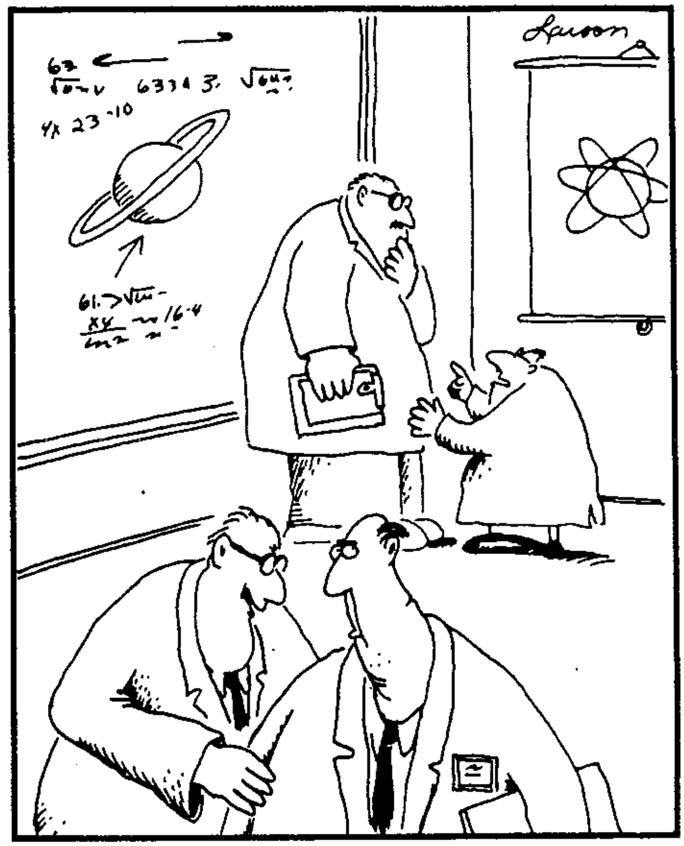
From a group of 8 men and 10 women, a committee of 5 is to be formed. How many committees can be formed if the committee meets the following conditions?

(15) All are men

(16) There are 3 men and 2 women

(17) There is 1 man and 4 women

(18) All are women



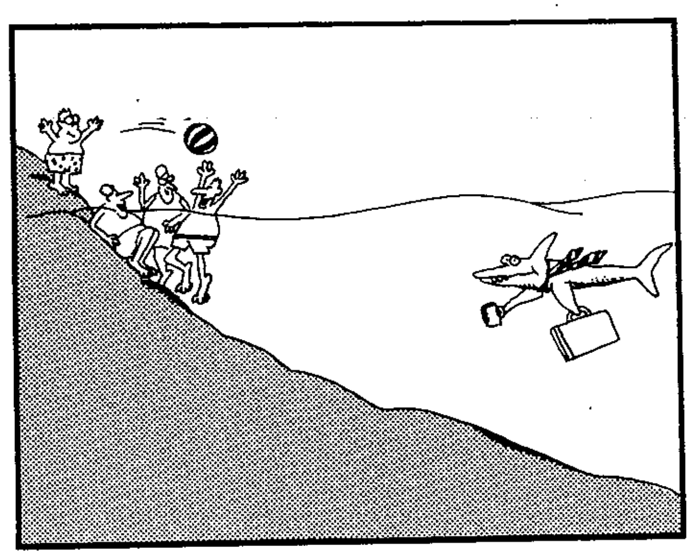
"There goes Williams again . . . trying to win support for his Little Bang theory."

Review

(19) How many different five-digit numbers between (not including) 82,000 and 87,999 can be made if no digit is repeated?

(20) How many different 8-letter patterns can be made from the letters in the word: PARALLEL ?

(21) How many ways can 7 keys be arranged on a ring with no clasp?



The shark on the go

Combinations & Permutations

UNIT 18 REVIEW & PRACTICE

Evaluate:

1. $P(11,4)$
2. $P(5,5)$
3. $C(9,3)$
4. $C(13,5)$

Solve:

5. A store sells seven different styles of gloves, each coming in a choice of four different colors. How many different gloves does the store sell?

6. How many ways can eight pictures be arranged in a straight line on the wall?

7. How many ways can eight students be assigned to a row of twelve lockers?

8. How many ways can the letters of the word "television" be arranged?

9. How many ways can seven people sit in a circle?

10. How many ways can nine different chairs be arranged around a circular table?

11. How many ways can eight keys be arranged on a key ring?

12. How many ways can six different color beads be arranged on a necklace with a clasp?

13. How many different starting units of five players can be selected from a team of twelve players?

14. How many math teams of 2 boys and 2 girls can be selected from an ATIM class with 10 boys and 8 girls?

15. There are five mystery books and seven romance novels on the reading list. How many lists of four mystery books and three romance novels can be made?

16. Eight cards are selected from a fifty-two card deck. How many different eight card combinations include two cards from each suit?

17. How many ways can 6 cards be drawn from a 52 card deck so that 3 are one suit and the remaining cards are from 2 other suits?

UNIT 19

Mathematics of Chance

Probability

DEMONSTRATION 19.1

When a coin is tossed, only two outcomes are possible: heads or tails. The desired outcome is called a success. The other outcome is a failure.

If an event can succeed "s" ways and fail "f" ways:

Probability:		Odds:
$P(s) = \frac{s}{s+f}$	$P(f) = \frac{f}{s+f}$	Odds of success = ratio s to f

① A bag contains 5 blue and 4 white marbles. If one marble is chosen at random, what is the probability it is blue?

$$\frac{5}{5+4} = \frac{5}{9}$$

② A committee of 2 is to be selected from 6 men and 3 women. What is the probability of selecting 2 women?

$$\frac{C(3,2)}{C(9,2)} = \frac{\frac{3!}{1!2!}}{\frac{9!}{7!2!}} = \frac{1}{12}$$

③ What are the odds of tossing a die and getting a 3?

ratio of s to f: 1 to 5

④ If 5 cards are drawn from a 52 card deck, what are the odds that 4 will be from the same suit?

Probability:

$$\frac{P(4,2) \cdot C(13,4) \cdot C(13,1)}{C(52,5)}$$

$$\frac{\frac{4!}{2!} \cdot \frac{13!}{9!4!} \cdot \frac{13!}{12!1!}}{\frac{52!}{47!5!}} = \frac{111{,}540}{2{,}598{,}960}$$

$$\frac{143}{3332}$$

odds: $3332 - 143 = 3189$

143 to 3189

Probability

PROBLEM SET 19.1

In a bag of 7 pennies, 4 nickels, and 5 dimes – 3 coins are selected at random. Determine the probability that:

1. all are pennies
2. all are dimes
3. 2 are pennies, 1 is a dime
4. one of each coin is drawn

Suppose you select 2 letters from the word "algebra". what is the probability of selecting:

5. 1 vowel and 1 consonant
6. 2 vowels
7. 2 consonants

There are 8 mystery books and 9 science fiction books. 4 are selected.

Determine the odds of choosing:

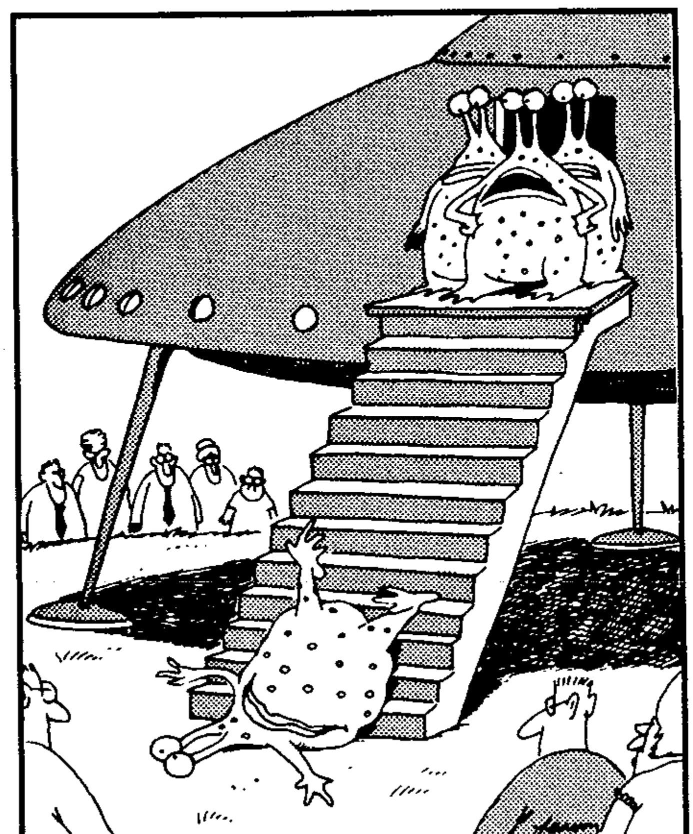
8. 4 mystery books
9. 2 of each

Determine the probability of selecting:

10. 4 science fiction books
11. 3 mysteries, 1 science fiction

From a deck of 52 cards, 5 are dealt. Determine the odds of dealing:

12. 5 aces
13. 5 face cards
14. 3 from one suit, 2 from another
15. 5 from one suit



"Wonderful! Just wonderful! . . . So much for instilling them with a sense of awe."

Independence & Exclusivity

DEMONSTRATION 19.2

If two events are independent, the probability they will both occur is the product of their probabilities:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

If events are dependent, the probability they will both occur is the product of one probability and the other's dependent probability:

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$$

① There are 5 nickels, 7 dimes, and 9 pennies in a coin purse. If two coins are selected, what is the probability of selecting a penny followed by a dime – with and without replacement?

With replacement (Ind.)

$$\frac{9}{21} \cdot \frac{7}{21} = \boxed{\frac{1}{7}} \approx .143$$

Without replacement (Dep.)

$$\frac{9}{21} \cdot \frac{7}{20} = \boxed{\frac{3}{20}} = .15$$

If two events cannot occur simultaneously, they are considered to be mutually exclusive:

$$P(A \text{ or } B) = P(A) + P(B)$$

If events are inclusive: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

② From the coin purse in problem #1, what is the probability of selecting a dime <u>or</u> a nickel if one coin is selected?

$$\frac{7}{21} + \frac{5}{21} = \frac{12}{21} = \boxed{\frac{4}{7}} \approx .571$$

Independence & Exclusivity

DEMONSTRATION 19.2

(3) Two cards are selected from a regulation deck. What is the probability that they are both kings or both black?

kings: black: black kings:

$$\frac{4}{52}\cdot\frac{3}{51} + \frac{26}{52}\cdot\frac{25}{51} - \frac{2}{52}\cdot\frac{1}{51} = \frac{660}{2652} = \boxed{\frac{55}{221}} \approx .249$$

(4) Three cards are selected from a regulation deck. What is the probability they are all red or all face cards?

red: face cards: red face cards:

$$\frac{26}{52}\cdot\frac{25}{51}\cdot\frac{24}{50} + \frac{12}{52}\cdot\frac{11}{51}\cdot\frac{10}{50} - \frac{6}{52}\cdot\frac{5}{51}\cdot\frac{4}{50} = \boxed{\frac{28}{221}} \approx .127$$

Sometimes probabilities include combinations:

(5) There are 3 black, 2 white, and 5 green marbles in a bag. What is the probability of selecting 3 marbles of the same color:

$$\frac{C(5,3)+C(3,3)}{C(10,3)} = \frac{\frac{5!}{2!3!}+\frac{3!}{0!3!}}{\frac{10!}{7!3!}} = \boxed{\frac{11}{120}} \approx .09$$

What is the probability of selecting <u>at least</u> 1 black marble?

$$\frac{C(3,3)+C(3,2)\cdot C(7,1)+C(3,1)\cdot C(7,2)}{C(10,3)} = \boxed{\frac{17}{24}} \approx .708$$

(6) A committee of 5 people is randomly selected from a group of 7 men and 6 women. What is the probability of selecting at least 3 women?

$$\frac{C(6,5)+C(6,4)\cdot C(7,1)+C(6,3)\cdot C(7,2)}{C(13,5)} = \frac{177}{429} = \boxed{\frac{59}{143}} \approx .413$$

Independence & Exclusivity

PROBLEM SET 19.2

The letters A, B, E, I, J, K, and M are written on cards and placed in a box. Two letters are selected. What is the probability of selecting:

1. two vowels, no replacement
2. two vowels, with replacement
3. two consonants, no replacement

There are 6 plates, 5 saucers, and 5 cups on the counter. Charlie accidentally knocks off two and breaks them. What is the probability that he broke:

4. two plates
5. a cup and saucer in that order
6. a cup and saucer in any order

A red and a green die are tossed. What is the probability that:

7. neither show 3
8. red shows 3, green shows 4
9. red shows 3, green does not show 3
10. both show different numbers

In a bag are 6 red, 5 white marbles. Three are selected. What is the probability of selecting:

11. less than two white
12. at least two white marbles

Two cards are drawn from a standard deck. What is the probability of drawing:

13. both aces or both face cards
14. both 7's or both red
15. both black or both face cards
16. both are 3's, 10's, or red

From a group of 6 men and 8 women, 6 people volunteer for a committee. What is the probability that the volunteers are:

17. all men or all women
18. more than four women
19. at least five men or fewer than two men

Binomial Trials

DEMONSTRATION 19.3

Problems that can be solved using a binomial expansion are called binomial trials. A binomial trial exists if and only if the following conditions occur:

1. There are only two possible outcomes
2. The events are independent

① Arthur normally wins 1 out of every 3 backgammon games he plays. What is the probability he will win 3 and lose 1?

4 games: Probability of $W = 1/3$ Probability of $L = 2/3$

$$(W+L)^4 = W^4 + 4W^3L + 6W^2L^2 + 4WL^3 + L^4$$

coefficient	term	meaning
C (4,4)	W^4	1 way to win all 4 games
C (4,3)	$4W^3L$	4 ways to win 3 and lose 1
C (4,2)	$6W^2L^2$	6 ways to win 2 and lose 2
C (4,1)	$4WL^3$	4 ways to win 1 and lose 3
C (4,0)	L^4	1 way to lose all 4 games

Use the 2nd term: $4W^3L = 4\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = 4\left(\frac{1}{27}\right)\left(\frac{2}{3}\right) = \boxed{8/81} \approx .099$

② When 5 coins are tossed, what is the probability of 3 heads and 2 tails?

$$C(5,3)\,H^3T^2 = 10\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 = 10\left(\tfrac{1}{8}\right)\left(\tfrac{1}{4}\right) = 10/32 = \boxed{5/16} \approx .313$$

5 coins

3 heads

Probability of heads

Probability of tails

Binomial Trials

DEMONSTRATION 19.3

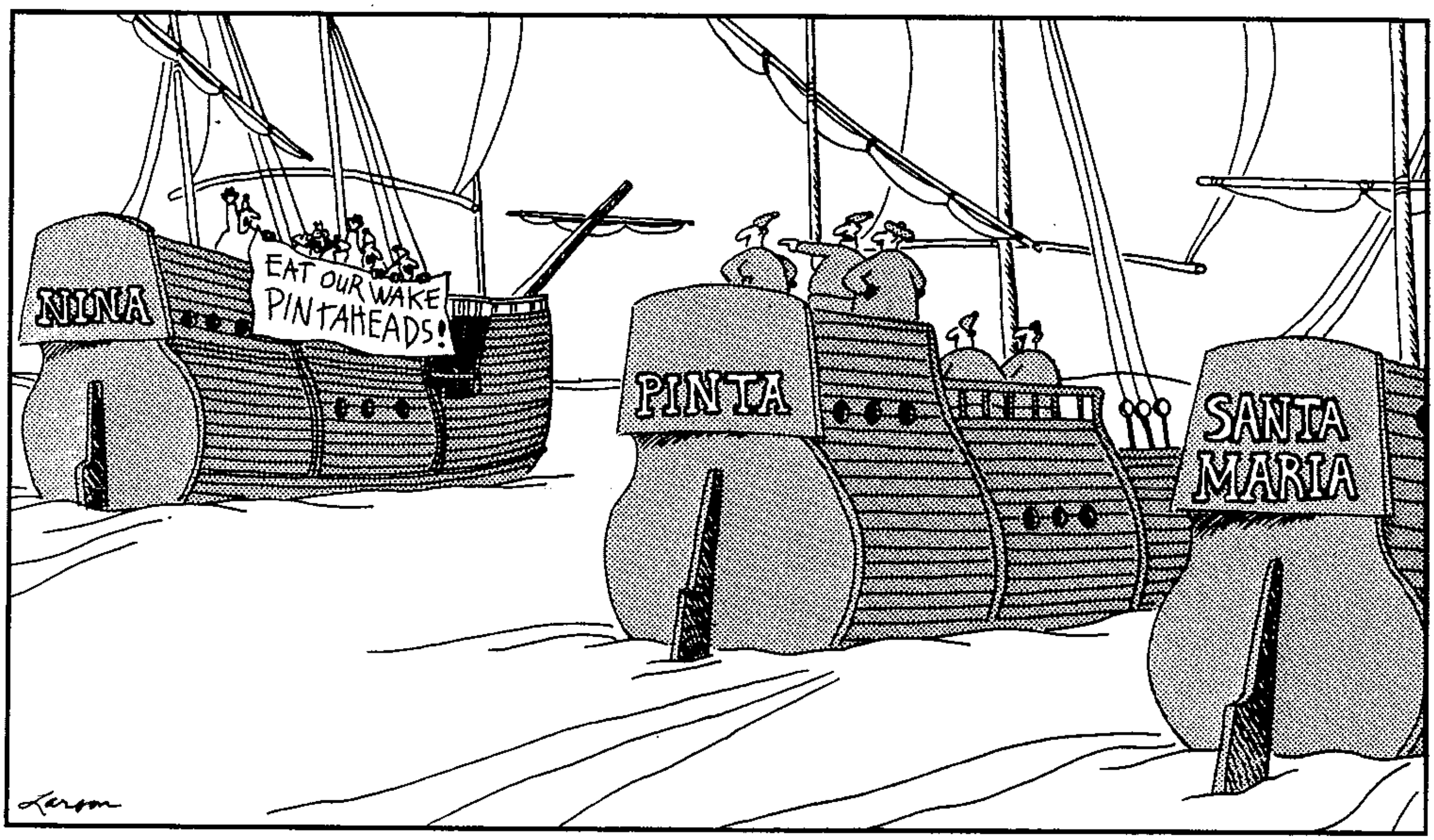
(3) Amy and Marla play 7 games. The probability that Amy wins a game is 1/5. What is the probability that Amy will win at least 3 games?

$$C(7,7)A^7 + C(7,6)A^6M + C(7,5)A^5M^2 + C(7,4)A^4M^3 + C(7,3)A^3M^4$$

$$A^7 + 7A^6M + 21A^5M^2 + 35A^4M^3 + 35A^3M^4$$

$$\left(\tfrac{1}{5}\right)^7 + 7\left(\tfrac{1}{5}\right)^6\left(\tfrac{4}{5}\right) + 21\left(\tfrac{1}{5}\right)^5\left(\tfrac{4}{5}\right)^2 + 35\left(\tfrac{1}{5}\right)^4\left(\tfrac{4}{5}\right)^3 + 35\left(\tfrac{1}{5}\right)^3\left(\tfrac{4}{5}\right)^4$$

$$\frac{1 + 28 + 336 + 2240 + 8960}{78{,}125} = \boxed{\frac{2313}{15{,}625}} \approx .148$$



Binomial Trials

PROBLEM SET 19.3

Use binomial trials to determine the probability:

Cathy Black has a bent coin. The probability of heads is $2/3$ with this coin. She flips the coin 4 times. What is the probability of:

1. no heads
2. at least 3 heads
3. no more than 2 heads

A batter with a .200 average comes to bat 5 times during a game. Determine the probability:

4. exactly 3 hits
5. at least 4 hits
6. at least 2 hits

If a tack is dropped, the probability it will land point up is $2/5$. Ten tacks are dropped. Determine the probability of the following:

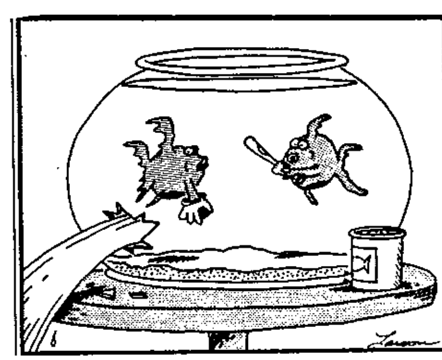
7. all land point up
8. exactly 5 land point up
9. at least 6 point up

Review

10. Three cards are drawn from a standard deck. What are the odds that 2 are from one suit and 1 from another?

11. There are 6 pennies and 4 nickels in a cup. If 4 coins are selected at random, find the probability of selecting 3 nickels and 1 penny.

12. There are 5 boys and 4 girls in the club. 2 of the boys and 2 of the girls are new club members. If the club leader selects two members at random, what is the probability those members are either boys or new members?



Mathematics of Chance

UNIT 19 REVIEW & PRACTICE

In a bag are 4 black, 3 white, and 3 green marbles. With no replacement, what is the probability of selecting:

1. 3 marbles — all white
2. 3 marbles — all black or all green
3. 4 marbles — at least 3 black

Four cards are drawn from a standard deck. Determine the probability of drawing:

4. All four face cards
5. A red card or a face card on the first draw
6. Three from one suit

A coin is tossed 7 times. Determine the probability of:

7. 5 heads
8. At least 4 tails
9. 3 heads or 4 heads

A.J. makes 3 out of every 5 free throws he attempts during the basketball season. If he attempts 8, determine the probability he will:

10. Make 6 of them
11. Make at least 7 of them
12. Make more than 7 or less than 2

Determine the probability of drawing:

13. A black marble followed by a green marble from the bag in problems 1-3.
14. Two even number cards or two red cards from a standard deck.

Determine the odds of:

15. Rolling an odd number twice in a row with a standard die.